CO 446 Matroid Theory

Ilia Chtcherbakov

Spring 2015

0 Preamble

CO 446, Matroid Theory. Offered by Professor Jim Geelen at the University of Waterloo in the Spring 2015 term. Composed, mlbaked by Ilia Chtcherbakov (`ischtche@uwaterloo.ca`). Special thanks to Kevin "Pure" Matthews.

Textbook: J.G. Oxley. *Matroid Theory, 2 ed.* New York: Oxford University Press, 2011. ISBN 978-0-19-960339-8.

0.1 Introduction

Matroids are in some sense a generalization of the notion of linear independence. This course is an introduction to matroids, with most of its focus on graphic matroids and the graph-theoretic extensions and implications associated to them: planarity, colouring, matchings, and connectivity.

The topics covered include extremal matroid theory, matroid intersection and partition, graphic matroids, matroid connectivity, and binary matroids. As well, there will be analogues of Turán's Theorem, Ramsey's Theorem, and the Erdős-Stone Theorem in binary projective geometries.

These notes will not deviate significantly from the narrative presented in class, except perhaps in organization. Exercises tend to be used as assignment questions. Sections marked with $(*)$ are difficult, possibly because machinery developed later will make them easier. These notes are complete. Latest update: 2 June 2016.

1 Matroids

4 May A **matroid** is defined as a pair $(E, \mathscr{I})$, where E is a finite **ground set** and $\mathscr{I}$ is a collection of **independent subsets** of E, satisfying the following properties:

(I1) the empty set is independent,

(I2) subsets of independent sets are independent,

(I3) the maximal independent subsets of any subset of E all have the same size.

If $M = (E, \mathscr{I})$ is a matroid, say that the ground set $E(M)$ is E, the collection of independent subsets $\mathscr{I}(M)$ is $\mathscr{I}$, the **size** $|M| := |E|$ is the size of the ground set, and the **rank** $r(M) := \max\{\, |I| \mid I \in \mathscr{I} \,\}$ is the size of the largest independent set.

There are a multitude of ways to both define and obtain matroids. Below are some examples of matroids.

Given a graph $G = (V, E)$, the **cycle matroid** of G is $M(G) := (E, \mathcal{F})$, where $\mathcal{F} = \{\, F \subseteq E \mid F \text{ is a forest} \,\}$. A matroid is **graphic** if it is the cycle matroid of some graph.

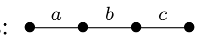
Given a matrix $A \in \mathbb{F}^{r \times E}$, the **column matroid** of A is defined as $M(A) := (E, \mathscr{I})$, where

$$\mathscr{I} = \{\, I \subseteq E \mid \text{the } I \text{ columns of } A \text{ are independent} \,\}.$$

A matroid is **$\mathbb{F}$-representable** if it is the column matroid of some matrix over $\mathbb{F}$. If $\mathbb{F} = \mathrm{GF}(2)$, the matroid is **binary**. As will be seen later, graphic matroids are representable in any field.

Given a graph G, let $\mathscr{M}$ be the set of matchings on G. Is $(E(G), \mathscr{M})$ a matroid?

Claim. *No.*

Proof. $(E(G), \mathscr{M})$ fails to satisfy property (**I3**) in general. Let G be the path on four vertices: 

. Fixing $E(G)$ as our subset, we see the matchings $\{ a, c \}$ and $\{ b \}$ are maximal but have different sizes. □

In some sense, matchings would not be difficult if they made a matroid as above. There is, however, a related construction. Given a graph G, say that $U \subseteq V(G)$ is **matchable** if there exists a matching M on G such that every vertex of U is incident to an edge of M. Then we can define the **matchable subset matroid**

$$\mathrm{MSM}(G) := (V(G), \{ U \subseteq V(G) \mid U \text{ is matchable} \}).$$

Showing that this is a matroid is left as an exercise to the reader.

Given a size $n \geq 0$ and a rank $0 \leq r \leq n$, the **uniform matroid** is defined by $U_{r,n} := (E, \mathscr{I})$, where

$$E = \{ 1, ..., n \} \quad \text{and} \quad \mathscr{I} = \{ X \subseteq E \mid |X| \leq r \}.$$

Every $U_{r,n}$ is $\mathbb{R}$-representable by an $r \times n$ matrix, with the intuition being analogous to finding a set of n points in $\mathbb{R}^r$ such that every subset of r points is in general position—no three colinear, no four coplanar, etc.

Lemma. *$U_{2,n}$ is $\mathbb{F}$-representable iff $|\mathbb{F}| \geq n - 1$.*

Proof. Suppose $U_{2,n}$ is $\mathbb{F}$-representable. Then there exists an $A \in \mathbb{F}^{2\times n}$ such that $U_{2,n} = M(A)$. Row operations and nonzero column scaling don't affect linear independence of the columns of a matrix, so WLOG A can be written

$$A = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & a_2 & a_3 & \dots & a_n \end{bmatrix},$$

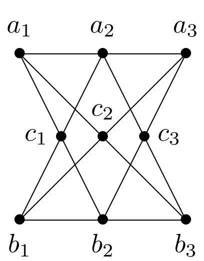
where the $a_2, ..., a_n$ are distinct elements of $\mathbb{F}$. Hence $|\mathbb{F}| \geq n - 1$. The converse is similar. □

6 May This leads right into an open problem: *given a finite field $\mathbb{F}$, which uniform matrices are $\mathbb{F}$-representable?*

Conjecture. *$U_{r,n}$ is $\mathbb{F}$-representable iff $r \in \{ 0, 1, n, n-1 \}$; or $n \leq |\mathbb{F}| - 1$; or $|\mathbb{F}|$ is even, $r \in \{ 3, n-3 \}$, and $n = |\mathbb{F}| + 2$.*

The "if" direction is known to hold. The "only if" direction was developed for $r \leq 5$ in the period between 1947–1970, and recently it was shown for $|\mathbb{F}|$ prime as well (Ball 2010).

Exercise $(*)$. Show that if $|\mathbb{F}|$ is odd and $n \geq |\mathbb{F}| + 2$, then $U_{3,n}$ is not $\mathbb{F}$-representable.



The non-Pappus configuration.

Every matroid seen thus far is representable, so we now present a famous example of one that is not. Let $E(M) = \{ a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \}$ and say that $I \in \mathscr{I}(M)$ iff $|I| \leq 3$ and no triple of distinct points in I shares a line in the configuration of points and lines at left. Then M is a matroid called the **non-Pappus matroid**.

Theorem. *M is not representable over any field.*

The proof is left as an exercise, but the result stems from Pappus of Alexandria's hexagon theorem, stating that if there are two triples of colinear points, as with $\{ a_1, a_2, a_3 \}$ and $\{ b_1, b_2, b_3 \}$ in the diagram, then geometrically, the triple $\{ c_1, c_2, c_3 \}$ found at the marked intersections must also be colinear. Trying to represent the non-Pappus configuration with a matrix falls prey to Pappus' result, since $\{ c_1, c_2, c_3 \}$ really want to be colinear in spaces over a field.

However, this non-representable behaviour is nothing special, as matroids go.

Theorem (Nelson 2016). *The proportion of representable n-element matroids tends to 0 as $n \to \infty$.*

The above remained a conjecture for a long time. What follows is a similar conjecture in "asymptotic matroid theory", which continues to elude us.

A matroid M is **paving** if all subsets of $E(M)$ of size strictly less than $r(M)$ are independent. Uniform matroids are paving, but it is not hard to find non-uniform examples: $M(K_4)$ is paving, where K_4 is the complete graph on four vertices.

Conjecture. *The proportion of paving n-element matroids tends to 1 as $n \to \infty$.*

1.1 Equivalent Definitions

Matroids have a wide variety of definitions and perspectives, each of which has its own merits.

A **circuit** in a matroid is a minimal dependent set. This is taken in analogy with circuits in graphs.[1] Circuits of size one are called **loops**. Two elements of the ground set which together form a circuit of size two are called **parallel**. A matroid is **simple** if it has no loops or parallel pairs.

Lemma. *Let $\mathscr{C}$ denote the collection of circuits in a matroid M. Then*

(**C1**) *the empty set is not in $\mathscr{C}$,*

(**C2**) *if $C, C' \in \mathscr{C}$ are distinct, then $C \nsubseteq C'$,*

(**C3**) *if $C, C' \in \mathscr{C}$ are distinct, and $e \in C \cap C'$, then there exists a distinct $D \in \mathscr{C}$ such that $D \subseteq C \cup C' - e$.*

Proof. (**C1**) and (**C2**) are trivial. Let $C, C' \in \mathscr{C}$ be distinct and take $e \in C \cap C'$. $C \cap C'$ is independent by (**C1**). Suppose for a contradiction that $I = C \cup C' - e$ is independent. With the intent of applying axiom (**I3**), fix the subset $C \cup C'$. Then I is maximal. Let I' be a maximal independent subset of $C \cup C'$ such that $C \cap C' \subseteq I'$. By (**I3**), $|I| = |I'|$, but to have that many elements either $C \subseteq I'$ or $C' \subseteq I'$, which contradicts the independence of I'. So I is not independent and thus contains a circuit. □

Theorem. *Let $\mathscr{C}$ be a collection of subsets of a finite ground set E. Then $\mathscr{C}$ is the collection of circuits of a matroid iff $\mathscr{C}$ satisfies* (**C1–3**). *Proof. Exercise.* □

8 May A **basis** of a matroid is a maximal independent subset. By (**I3**) every basis of M will have size $r(M)$.

Lemma. *Let $\mathscr{B}$ denote the collection of bases of a matroid M. Then*

(**B1**) *$\mathscr{B}$ is nonempty,*

(**B2**) *if $B, B' \in \mathscr{B}$ are distinct, and $e \in B - B'$, then there exists an $f \in B' - B$ such that $(B - \{e\}) \cup \{f\} \in \mathscr{B}$.*

Proof. (**B1**) is trivial. Let $B, B' \in \mathscr{B}$ be distinct, and take $e \in B - B'$. Intending to apply (**I3**), fix the subset $X := (B - \{e\}) \cup B'$. Each of $B - \{e\}$ and B' are independent subsets of X, and $|B - \{e\}| < |B'|$, so there exists an $f \in X - (B - \{e\}) = B' - B$ such that $(B - \{e\}) \cup \{f\}$ is a maximal independent subset of X. □

Theorem. *Let $\mathscr{B}$ be a collection of subsets of a finite ground set E. Then $\mathscr{B}$ is the collection of bases of a matroid iff $\mathscr{B}$ satisfies* (**B1**) and (**B2**). *Proof. Exercise.* □

Given $X \subseteq E(M)$, define the **rank function** of a matroid M by $r_M(X) = \max\{\, |I| \mid I \in \mathscr{I} \text{ and } I \subseteq X \,\}$.

Lemma. *Let M be a matroid and let $r = r_M$. Then*

(**R1**) *for all $X \subseteq E(M)$, $0 \le r(X) \le |X|$,*

(**R2**) *for all $X \subseteq Y \subseteq E(M)$, $r(X) \le r(Y)$,*

(**R3**) *for all $X, Y \subseteq E(M)$, $r(X) + r(Y) \ge r(X \cap Y) + r(X \cup Y)$—that is, r is* ***submodular***.

Proof. (**R1**) and (**R2**) are trivial. Let $X, Y \subseteq E(M)$, and take $I_\cap \in \mathscr{I}(M)$ maximal subject to $I_\cap \subseteq X \cap Y$. Likewise, take maximal independent I_X subject to $I_\cap \subseteq I_X \subseteq X$ and $I_\cup$ subject to $I_X \subseteq I_\cup \subseteq X \cup Y$. Finally, $I_Y := Y \cap I_\cup$. Observe that $|I_\cap| = r(X \cap Y)$, $|I_X| = r(X)$, and $|I_\cup| = r(X \cup Y)$, but also $|I_Y| \le r(Y)$. Then

$$r(X \cap Y) + r(X \cup Y) = |I_\cap| + |I_\cup| = |I_X| + |I_Y| \le r(X) + r(Y). \quad \square$$

Theorem. *Let E be a finite set and let $r : 2^E \to \mathbb{Z}$ be a function. Then r is the rank function of a matroid iff r satisfies* (**R1–3**). *Proof. Exercise.* □

There exists a fourth rank function condition, called *Ingleton's Inequality*. For any $X_1, X_2, X_3, X_4 \subseteq E(M)$,

$$r(X_1) + r(X_2) + r(X_1 \cup X_2 \cup X_3) + r(X_1 \cup X_2 \cup X_4) + r(X_3 \cup X_4) \le r(X_1 \cup X_2) + r(X_1 \cup X_3) + r(X_1 \cup X_4) + r(X_2 \cup X_3) + r(X_2 \cup X_4).$$

Ingleton has shown that all representable matroids satisfy this inequality, but that there exist matroids which do not satisfy it. It is one in a family of increasingly complicated rank inequalities refining representability.

[1] We opt to resolve naming conflicts in favour of established matroid terminology, deviating from the graph theoretic 'cycle'.

For a matroid M and $X, Y \subseteq E(M)$, define the **local connectivity** $\sqcap_M(X,Y) := r_M(X) + r_M(Y) - r_M(X \cup Y)$. The submodularity of the rank function may be equivalently stated $\sqcap_M(X,Y) \geq r_M(X \cap Y)$.

Lemma. *Let $A \in \mathbb{F}^{r \times E}$ and for each $X \subseteq E$ define $\overline{X}$ to be the subspace spanned by the columns of $A\big|X$.*[2] *Then for $X, Y \subseteq E$, $\dim(\overline{X} \cap \overline{Y}) = \sqcap_{M(A)}(X,Y)$.*

Proof. From linear algebra, $\dim(\overline{X}) + \dim(\overline{Y}) = \dim(\overline{X} \cap \overline{Y}) + \dim(\overline{X \cup Y})$. Considering the column matroid $M(A)$, $\dim(\overline{X}) = r(X)$, so $\dim(\overline{X} \cap \overline{Y}) = r(X) + r(Y) - r(X \cup Y) = \sqcap(X,Y)$. $\square$

While this is not a novel result, it unearths some intuition for local connectivity in representable matroids.

1.2 Geometry

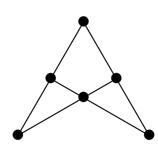
Lemma. *Let $X, X' \subseteq E(M)$ and $X_\cap \subseteq X \cap X'$. If $r(X) = r(X') = r(X_\cap)$ then $r(X \cup X') = r(X_\cap)$.*

Proof. By submodularity, $2r(X_\cap) = r(X) + r(X') \geq r(X \cap X') + r(X \cup X') \geq r(X_\cap) + r(X \cup X') \geq 2r(X_\cap)$. Thus equality holds at every step, and in particular $r(X \cup X') = r(X_\cap)$. $\square$

11 May By the above lemma, for any subset X of the ground set there is a unique maximal superset $\widehat{X} \supseteq X$ such that $r(\widehat{X}) = r(X)$. This is the **closure** $\mathrm{cl}_M(X)$ of X. Observe that an $e \in E(M) - X$ is in $\mathrm{cl}_M(X)$ iff there exists a circuit C such that $C - X = \{e\}$. Notice also that if $X \subseteq Y$, then $\mathrm{cl}(X) \subseteq Y$. Closures can be used to define a matroid as well, but we omit the construction and proof of equivalence.

A set $F \subseteq E(M)$ is a **flat** if $\mathrm{cl}_M(F) = F$. (*Exercise.* Show that that intersection of two flats is a flat.) A rank-1 flat is a **point**. A rank-2 flat is a **line**. A rank-3 flat is a **plane**. Also, a **hyperplane** is a flat of rank $r(M)-1$.

As implied by this terminology, there is a nice way to interpret simple matroids of small rank as diagrams and vice versa. A matroid M can be represented in $\mathbb{R}^{r(M)}$-space by giving its matroid points as geometric points, its matroid lines as geometric lines—not necessarily straight—crossing through all those points which have that flat as their pairwise closure, and so on.

$U_{2,4}$ $M(K_4)$ 

This is of course subject to some obvious geometric considerations: two lines can intersect in at most one point, if two planes intersect in more than one point it must be at a line, etc. If the matroid is simple, points can be taken as elements of the ground set; otherwise, they are simply the parallel classes, and loops are hidden entirely. This contextualizes the non-Pappus matroid defined earlier, if one takes all points to be on the same plane.

Let W_k denote the number of flats of rank k in a matroid M, taking $W_k = 0$ if $k \notin \{0, ..., r(M)\}$.

Conjecture (Rota 1971). $W_k \geq \min\{W_{k-1}, W_{k+1}\}$.

Points–Lines–Planes Conjecture (Mason 1972). *In a rank-4 matroid, $W_2^2 \geq W_1 W_3$.*

Taking a cue from how a matroid diagram "simplifies" a matroid by removing loops and collapsing parallel classes, we define a **simplification** of a matroid M. Let L be the collection of loops of M and $(P_1, ..., P_k)$ be the points of M. L is contained in each P_i, so consider $(P_1 - L, ..., P_k - L)$—it is a partition of $E(M) - L$ into parallel classes. Choosing an $e_i \in P_i$ for each i as a representative, the restriction of M to $\{e_1, ..., e_k\}$ is a simplification of M.

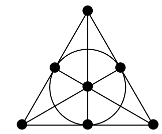
Exercise. Show that simplification is determined uniquely up to isomorphism, thus earning a definite article.

Exercise. Show that a matroid is $\mathbb{F}$-representable iff its simplification is.

Let r be a positive integer and $\mathbb{F}$ a finite field of order q. Let $A \in \mathbb{F}^{r \times q^r}$ be a matrix with distinct columns. $M(A)$ is not simple, as it contains a zero column, and also every nonzero multiple of every column it contains. Define the **projective geometry**—denoted $\mathrm{PG}(r-1, \mathbb{F})$ or $\mathrm{PG}(r-1, q)$—as the simplification of $M(A)$.

[2] This notation is to be understood as 'A restricted to X'.

Observe $|\mathrm{PG}(r-1,q)| = \frac{q^r-1}{q-1}$, and that every line has length $q+1$. A small example is $F_7 = \mathrm{PG}(2,2)$, the matroid at right, whose diagram is known as the *Fano plane*. Observe also that every simple $\mathbb{F}$-representable rank-r matroid is isomorphic to a restriction of $\mathrm{PG}(r-1,\mathbb{F})$. This illuminates an exercise presented earlier:



$F_7 = \mathrm{PG}(2,2)$

Exercise ($*$). Show that if $|\mathbb{F}|$ is odd and $n \geq |\mathbb{F}|+2$, then $U_{3,n}$ is not $\mathbb{F}$-representable.

Hint. Consider $U_{3,n}$ as a restriction of $\mathrm{PG}(2,\mathbb{F})$, and count its lines and intersections.

1.3 Duality

13 May Given a matroid $M = (E, \mathcal{I})$, define its **dual** $M^* := (E, \mathcal{I}^*)$ by taking $\mathcal{I}^* := \{\, I^* \subseteq E \mid r(E - I^*) = r(M)\,\}$.

Theorem. *M^* is a matroid.*

Proof. (**I1**) and (**I2**) are obvious. Fix a $X \subseteq E(M)$, let I^*, J^* be maximal M^*–independent subsets of X, and let I be a maximal M–independent subset of $E(M) - X$. Take bases B and C of M such that $I \subseteq B \subseteq E(M) - I^*$ and $I \subseteq C \subseteq E(M) - J^*$. Since I was maximal, $B - X = I = C - X$. Moreover, by the maximality of I^* and J^*, $I^* = X - B$ and $J^* = X - C$. Hence $|I^*| = |X - B| = |X - C| = |J^*|$, and M^* satisfies (**I3**). $\square$

The independent sets of M are called **co–independent sets** of M. A basis in M^*—called a **cobasis** of M—is simply the complement of a basis of M. It immediately follows that $M^{**} = M$. A circuit of M^*—a **cocircuit** of M—is a minimal dependent set of M^*, so with respect to M it must be the complement of a maximal subset of rank $r(M) - 1$, which is exactly the definition of a hyperplane of M. Most matroid terminology can be prefixed with 'co–' to refer to the corresponding notion of the dual. Abbreviate the corank function $r_{M^*} =: r^*_M$.

Observe that $r(M) + r(M^*) = |M|$. As well, for a subset $X \subseteq E(M)$, $r^*(X) = |X| - \big(r(M) - r(E(M) - X)\big)$.

We now consider the duals of representable matroids. Given a subspace $U \leq \mathbb{F}^E$, recall the definition of the **orthogonal space** $U^\perp := \{\, x \in \mathbb{F}^E \mid x^\top u = 0 \text{ for all } u \in U\,\}$. $\dim U + \dim U^\perp = |E|$, though it is not necessarily true that U and $U^\perp$ only intersect at 0 when over a finite field.

Theorem (1.17). *Let $A_1 \in \mathbb{F}^{r_1 \times E}$ and $A_2 \in \mathbb{F}^{r_2 \times E}$. If $\mathrm{Row}(A_1)^\perp = \mathrm{Row}(A_2)$ then $M(A_1)^* = M(A_2)$.*

Proof. It suffices to show that if B is a basis of $M(A)$, then $E - B$ is a basis of $M(A')$. By row operations and column reordering we can WLOG assume that $A_1 = [\, I\ A\,]$, where the first $|B|$ columns are the basis B. Consider the matrix $A_2' = [\, -A^\top\ I\,]$. Since $A_2'\big|(E - B) = I$, $E - B$ is a basis of $M(A_2')$.

$\mathrm{Row}(A_1)^\perp = \mathrm{Row}(A_2')$. (*Exercise.*) Then, $\mathrm{Row}(A_2') = \mathrm{Row}(A_1)^\perp = \mathrm{Row}(A_2)$, so $M(A_2') = M(A_2)$. $\square$

Corollary. *M is $\mathbb{F}$-representable iff M^* is.*

Unlike with representable matroids, the dual of a graphic matroid is not always graphic. However...

Theorem. *If G is a planar graph, then $M(G^*) = M(G)^*$.*

Recall that it can be said that for a planar graph G, $E(G) = E(G^*)$. Nonsimple graphs—which are necessary considerations when taking planar duals—must be modelled as (V, E, I) for a vertex set V, an edge set E, and an incidence relation $I \subseteq V \times E$; and then there is a natural identification of edges in a graph and its dual.

Lemma. *Given a planar graph G and $X \subseteq E(G)$, $G\big|X$ is a forest iff $G^* - X$ is connected.*

Proof (Sketch). If X contains a circuit, $\mathbb{R}^2 - X$ is not connected, so neither is $G^* - X$. If on the other hand $G\big|X$ is a forest, $\mathbb{R}^2 - X$ remains connected, and since walks in $\mathbb{R}^2 - X$ crossing edges not in $G\big|X$ can be identified with walks on $G^* - X$, $G^* - X$ is connected. $\square$

Proof of Theorem. By the Lemma, independent sets of $M(G)$ give complements that are spanning subgraphs of G^*, i.e. those complements have full rank and thus are independent in $M(G^*)^*$. Thus $M(G) = M(G^*)^*$. $\square$

15 May From these results we also have a tidy proof of Euler's Formula for planar graphs.

Euler's Formula. *Given a planar graph with vertex set V, edge set E, and face set F, $|E| = |V| + |F| - 2$.*

Proof. Letting T be the edge set of a spanning tree—so that $E-T$ is the edge set of a spanning tree of the dual—

$$|E| = |T| + |E-T| = |V| - 1 + |F| - 1. \quad \square$$

Claim. *$M(K_5)^*$ is not graphic.*

Proof. Suppose $M(K_5)^* = M(G)$ for some connected graph G. Then $|E(G)| = |E(K_5)| = 10$ and

$$|V(G)| = r(M(K_5)^*) + 1 = |E(K_5)| - r(M(K_5)) + 1 = 10 - 4 + 1 = 7.$$

$2|E(G)| < 3|V(G)|$, so by the handshake lemma there exists a vertex in G of degree at most two. So $M(G)$ has a cocircuit—and hence $M(K_5)$ has a circuit—of size at most two. But this is a contradiction, since K_5 is simple. $\square$

Exercise. Show that $M(K_{3,3})^*$ is not graphic.

These facts will be used later to give a matroid-theoretic characterization of planarity.

Theorem (Whitney). *G is a planar graph iff $M(G)^*$ is graphic.*

This is somewhat surprising, because somehow matroids—inherently geometrical objects—know something about planarity, which is a topological concern.

1.4 Minors

Given a subset $D \subseteq E(M)$, define the **deletion** $M\backslash D := (E-D, \mathcal{I}')$, where $\mathcal{I}' := \{\, I \in \mathcal{I}(M) \mid I \subseteq E - D \,\}$. Observe that $M\backslash D$ is a matroid, and if M is $\mathbb{F}$-representable then $M\backslash D$ is also. Further, for a graph G, $M(G)\backslash D = M(G\backslash D)$.

Dually, given $C \subseteq E(M)$, define the **contraction** $M/C := (M^*\backslash C)^*$. While an unwieldy definition, it is still immediately useful: M/C is a matroid, again preserving representability. It is harder to derive the correspondence with graph contraction for non-planar graphs, but we will see that, just as with deletion, $M(G)/C = M(G/C)$.

Lemma. *For $X \subseteq E(M) - C$, $r_{M/C}(X) = r_M(X \cup C) - r_M(C) = r_M(X) - \sqcap_M(X, C)$. Proof. Exercise.* $\square$

Exercise. Let $X, Y \subseteq E(M)$ and $X' \subseteq X$. Show that $\sqcap(X', Y) \le \sqcap(X, Y)$, with equality when $r(X') = r(X)$.

Lemma. *If $C \subseteq E(M)$ and I is a maximal independent subset of C, then $M/C = M\backslash(C - I)/I$.* $\square$

Say that subsets $X, Y \subseteq E(M)$ are **skew** iff $\sqcap_M(X, Y) = 0$.

Lemma. *$I \subseteq E(M) - C$ is an independent set of M/C iff I is M-independent and I is skew to C.*

Proof. I is M/C–independent if $|I| = r_{M/C}(I) = r_M(I) - \sqcap_M(I, C)$. Since both are nonnegative quantities, it must be that $r_M(I) = |I|$ and $\sqcap_M(I, C) = 0$. $\square$

Exercise. Let X and Y be disjoint. Show that X is skew to Y iff there is no circuit $C \subseteq X \cup Y$ such that both $C \cap X$ and $C \cap Y$ are nonempty.

Exercise. Prove that $M(G)/C = M(G/C)$ for any graph G.

If $C, D \subseteq E(M)$ are disjoint sets, then $M\backslash D/C$ is a **minor** of M. We will treat minors up to isomorphism—say that M has an N-minor if M has a minor that is isomorphic to N.

Lemma (Scum Theorem). *If N is a minor of M, there exists a partition (C, D) of $E(M) - E(N)$ such that C is independent, D is co-independent, and $N = M/C\backslash D$. Proof. Exercise.* $\square$

20 May Consider a representable matroid $M(A) = (E, \mathcal{I})$. Let $C, D \subseteq E$ be disjoint subsets such that C is independent and D is coindependent. There exists a basis B such that $C \subseteq B \subseteq E - D$. By row operations we can assume that the first B columns of A are the identity. Supposing A is in the form at right, we find that $M(A)/C\backslash D = M\big([\,I \ A_1\,]\big)$.

$$\begin{array}{cc} & \begin{array}{cccc} \scriptstyle B-C & \scriptstyle C & \scriptstyle E-B\cup D & \scriptstyle D \end{array} \\ \begin{array}{c} \scriptstyle B-C \\ \scriptstyle C \end{array} & \left[\begin{array}{cccc} I & 0 & A_1 & A_2 \\ 0 & I & A_3 & A_4 \end{array}\right] \end{array}$$

A matroid is **regular** if it is representable over every field. Recall that a matrix is called **totally unimodular** if the determinant of every square submatrix is 0 or ± 1—these matrices commonly arise from graphs.

Theorem (Tutte). *Let M be a matroid. The following are equivalent:*

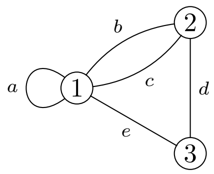
(a) *M is regular.*

(b) *M is representable over* GF(2) *and* GF(3).

(c) *$M = M(A)$ for a totally unimodular matrix A over $\mathbb{R}$.*

Proof. By excuse. (a) $\Rightarrow$ (b) is trivial; (b) $\Rightarrow$ (c) is straightforward but long; (c) $\Rightarrow$ (a) is obvious. $\square$

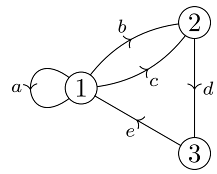
Given a graph G, fix some arbitrary orientation of the edges. A **signed incidence matrix** of G is a $V \times E$ matrix with (v, e)-entry 1 if, in the orientation, v is the head of e, -1 if v is the tail, and 0 if it is both or neither. The **unsigned incidence matrix** of G is a $V \times E$ matrix with (v, e)-entry equal to the number of times e is incident at v.



G

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

unsigned incidence matrix



orientation of G

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

signed incidence matrix

Observe that over GF(2), the signed and unsigned incidence matrices coincide.

Lemma. *Let A be a signed incidence matrix of a graph G and $\mathbb{F}$ be a field. Then $r(M_{\mathbb{F}}(A)) = |V(G)| - \operatorname{comps}(G)$.*

Proof. Consider $x \in \mathbb{F}^{V(G)}$. $x^{\mathsf{T}} A = 0$ iff $x_u = x_v$, for all vertices u, v sharing a connected component. Thus

$$r(M_{\mathbb{F}}(A)) = \operatorname{rank}(A) = |V(G)| - \operatorname{comps}(G). \quad \square$$

Lemma. *Let A be a signed incidence matrix of a graph G and let $\mathbb{F}$ be a field. Then $M_{\mathbb{F}}(A) = M(G)$.*

Proof. Apply the previous Lemma to $G|X$ for each $X \subseteq E(G)$. $\square$

Theorem. *Graphic matroids are regular.* $\square$

Exercise. Show that signed incidence matrices are totally unimodular.

There is an interesting combinatorial application of this machinery, leading to a natural proof of the Matrix–Tree Theorem. Consider the problem of counting the number of bases of $M(A)$, given a totally unimodular matrix A.

Cauchy–Binet Formula. *Let R, E be finite ordered sets and let $A, B \in \mathbb{F}^{R \times E}$. Then*

$$\det AB^{\mathsf{T}} = \sum_{X \in \binom{E}{R}} \big(\det A|X\big)\big(\det B|X\big),$$

where the sum is ranging over all subsets $X \subseteq E$ with $|X| = |R|$.

Proof (Sketch). By row operations, $\det \left[\begin{smallmatrix} I & B^{\mathsf{T}} \\ -A & 0 \end{smallmatrix}\right] = \det \left[\begin{smallmatrix} I & B^{\mathsf{T}} \\ 0 & AB^{\mathsf{T}} \end{smallmatrix}\right] = \det AB^{\mathsf{T}}$. Further, if $C, D \in \mathbb{F}^{V \times V}$ and D is diagonal, then it can be shown inductively that

$$\det(C + D) = \sum_{X \subseteq V} \big(\det C|(V - X)\big)\big(\det D|X\big).$$

The result follows by taking $\det\big(\left[\begin{smallmatrix} 0 & B^{\mathsf{T}} \\ -A & 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} I & \\ & 0 \end{smallmatrix}\right]\big) = \sum_{X \subseteq E} \det\big(\left[\begin{smallmatrix} 0 & B^{\mathsf{T}} \\ -A & 0 \end{smallmatrix}\right]|X\big)$. $\square$

Theorem. *If $A \in \{0, \pm 1\}^{R \times E}$ is totally unimodular and has full rank, the number of bases of $M(A)$ is* $\det AA^{\mathsf{T}}$.

Proof. $\det AA^{\mathsf{T}} = \sum_X \big(\det A|X\big)^2$—ranging over $X \in \binom{E}{R}$—but observe that $\big(\det A|X\big)^2$ is 1 iff X is a basis. $\square$

22 May Let A be the signed incidence matrix of a graph G, and define the **Laplacian** of G by $L := AA^{\mathsf{T}}$. Observe that $L_{uu} = \deg(u)$, and $-L_{uv}$ is the number of edges from u to v in G when $u \neq v$.

Corollary (Matrix–Tree Theorem). *Let $u \in V(G)$ and suppose $L_{\bar{u}}$ is obtained from L by deleting the u row and column. Then $\det L_{\bar{u}}$ is the number of spanning trees of G.* $\square$

Exercise. Let $u, v \in V(G)$ and suppose $L_{\bar{u},\bar{v}}$ is obtained from L by deleting the u row and the v column. Then $\det L_{\bar{u},\bar{v}}$ is the number of spanning trees of G.

A class $\mathscr{M}$ of matroids—or, more generally, any kind of object equipped with minors—is **minor-closed** if for all $M \in \mathscr{M}$, every minor of M is also in $\mathscr{M}$. Some exampled of minor-closed classes of graphs are $\mathbb{F}$-representable matroids, graphic matroids, regular matroids, uniform matroids, and paving matroids. Say that $M \notin \mathscr{M}$ is an **excluded minor** of $\mathscr{M}$ if every proper minor of M is in $\mathscr{M}$.[3]

Exercise. Determine the excluded minors of the class of uniform matroids.

Recall from graph theory Kuratowski's Theorem—a graph is planar iff it has no minor isomorphic to K_5 or $K_{3,3}$. This is a statement about the excluded minors of the class of planar graphs:

Kuratowski's Theorem. *The excluded minors of the class of planar graphs are K_5 and $K_{3,3}$.*

Observe that the class of graphic matroids is a subclass of the regular matroids, and that a subclass of the binary matroids.

Tutte's Excluded Minors Theorem.

(1) *The only excluded minor of the binary matroids is $U_{2,4}$.*

(2) *The excluded minors of the regular matroids are $U_{2,4}$, F_7, and F_7^*.*

(3) *The excluded minors of the graphic matroids are $U_{2,4}$, F_7, F_7^*, $M(K_5)^*$, and $M(K_{3,3})^*$.*

In this course we will only prove (1) and (3). (2) has a reasonably short proof due to Gerards. Notice that (3) implies Kuratowski's Theorem, in conjunction with a theorem of Whitney stated earlier.

Conjecture (Rota). *For each finite field $\mathbb{F}$, there are finitely many excluded minors for $\mathbb{F}$-representability.*

Rota's Conjecture is claimed to be true by Geelen, Gerards, and Whittle as of 2013. Also, it is known to be false for infinite fields (Lazarson).

Graph Minors Theorem. *Every minor-closed class of graphs has only finitely many excluded minors.*

This is equivalent to and sometimes known as the *well-quasi-ordering theorem*: in any infinite sequence $\{ G_i \}_{i=1}^{\infty}$ of finite graphs, there exists a graph G_i which is a minor of some G_j with $i < j$.

The same is not true of matroids. The collection of excluded minors for representability in any finite field is a counterexample, according to Lazarson. Even something like $\{ \mathrm{PG}(2,p) \mid p \text{ prime} \}$ forms an "infinite antichain" with respect to minors. In some sense, this means there are only countably many minor-closed classes of graphs, but uncountably many such classes of matroids.

1.5 Application: Matchings

Given a graph G, define the **matching number** $\nu(G)$ be the size of a maximum matching of G, the **deficiency** $\mathrm{def}(G) = |V(G)| - 2\nu(G)$, and denote the number of components of odd size in G by $\mathrm{odd}(G)$. Notice that $\mathrm{odd}(G) \geq \mathrm{def}(G)$, and that for any subset $X \subseteq E(G)$, $\mathrm{def}(G) \geq \mathrm{def}(G - X) - |X|$. So we have that each subset $X \subseteq E(G)$ gives a lower bound on the deficiency by the relatively simple $\mathrm{odd}(G - X) - |X|$.

Tutte-Berge Formula. $\mathrm{def}(G) = \max \{ \mathrm{odd}(G - X) - |X| \mid X \subseteq E(G) \}$.

25 May Say that a vertex $v \in V(G)$ is **avoidable** if $\nu(G - v) = \nu(G)$, and observe that $\mathrm{def}(G - v) = \mathrm{def}(G) - 1$ if v is avoidable, while $\mathrm{def}(G - v) = \mathrm{def}(G) + 1$ if it is not. Say that a graph G is **hypomatchable** if it is connected and every vertex is avoidable.

Lemma (Gallai). *If G is hypomatchable, then $\mathrm{def}(G) = 1$ and thus $|V(G)|$ is odd.*

[3] Excluded minors are exclusively considered up to isomorphism.

For a matroid M, we call $e \in E(M)$ a **coloop** if it is a loop of M^*—that is, $r(E - \{e\}) < r(M)$. Similarly, define a **series pair** to be the dual of a parallel pair: $\{ e, f \}$ are in series if $r(E - \{e\}) = r(E - \{f\}) = r(M)$, but $r(E - \{ e, f \}) < r(M)$.

Proof of Lemma. Recall the matchable subset matroid $\mathrm{MSM}(G) = (V, \text{matchable})$. Each vertex of G is avoidable by a maximum matching, so it is in some cobasis of $\mathrm{MSM}(G)$, and thus no vertex is a coloop. For any edge $uv \in E(G)$, $\nu(G - \{ u, v \}) < \nu(G)$, so $\{ u, v \}$ is a series pair, and since G is connected, by transitivity, each pair of vertices is series in $\mathrm{MSM}(G)$. Thus $\mathrm{MSM}(G) \cong U_{|V|-1,|V|}$ (*Exercise*.) and we see $\mathrm{def}(G) = 1$. □

Proof of Tutte–Berge. For each $X \subseteq V(G)$, $\mathrm{def}(G) \geq \mathrm{def}(G - X) - |X| \geq \mathrm{odd}(G - X) - |X|$, so it remains to find a subset at which equality holds. Choose maximal $X \subseteq V(G)$ such that $\mathrm{def}(G) = \mathrm{def}(G - X) - |X|$, so that $G - X$ is hypomatchable. Each component is hypomatchable, so $\mathrm{def}(G - X) = \mathrm{odd}(G - X)$. □

2 Binary Matroids

Binary matroids—matroids representable over GF(2)—are particularly nice matroids when it comes to representability. This is because, in determining a matrix representation of the matroid, it only matters whether an entry is zero or nonzero, since there is only one nonzero element of GF(2).

Let B be a basis of the matroid $M = (E, \mathcal{I})$. Define $F \in \mathrm{GF}(2)^{B \times (E - B)}$ by

$$F_{ef} := \begin{cases} 1 & (B - \{e\}) \cup \{f\} \text{ is a basis,} \\ 0 & \text{otherwise.} \end{cases}$$

This is the **fundamental matrix** of (M, B)—denote it by $F(M, B)$.

For $e \in B$, the **fundamental cocircuit** of e in (M, B) is the unique cocircuit disjoint from $B - \{e\}$. This is exactly the cocircuit $E - \mathrm{cl}(B - \{e\})$.

Exercise. Prove that the fundamental cocircuit of e in (M, B) is equal to $\{e\} \cup \{ f \in E - B \mid F(M, B)_{ef} = 1 \}$.

By duality, for $f \in E - B$, there exists a unique circuit in $B \cup \{f\}$—this is the **fundamental circuit** of f in (M, B). By the exercise above, this circuit is equal to $\{f\} \cup \{ e \in B \mid F(M, B)_{ef} = 1 \}$.

Lemma. *Let B be a basis of a binary matroid $M(A)$ and let $F = F(M(A), B)$. Then* $\mathrm{Row}(A) = \mathrm{Row}([\, I \;\, F \,])$.

Proof. Up to row operations, we may assume $A = [\, I \;\, A' \,]$. Then for each $e \in B$ and $f \notin B$, $(B - \{e\}) \cup \{f\}$ is a basis iff $A'_{ef} = 1$. Thus $A' = F$ by definition. □

In fact, we can show something stronger. The fundamental matrix is, in a precise sense, the blueprint of what must be zero or nonzero in a representation of a matroid. For binary matroids, when something is nonzero, we know exactly what it is.

Lemma. *Let B be a basis of M. Then M is binary iff $M = M\big([\, I \;\, F(M, B) \,]\big)$.* □

Despite the theoretical beauty of this result, checking the above given a rank oracle for a matroid is $\mathsf{EXPTIME}$.

27 May **Lemma** (2.3). *Suppose that $M(A)$ is a binary matroid and $M(A)^* = M(A')$. Then* $\mathrm{Row}(A)^\perp = \mathrm{Row}(A')$.

Proof. Observe that $F(M(A'), E - B) = F(M(A), B)^\mathsf{T}$. Letting $F = F(M(A), B)$, we have, by an earlier exercise, $\mathrm{Row}(A)^\perp = \mathrm{Row}([\, I \;\, F \,])^\perp = \mathrm{Row}([\, F^\mathsf{T} \;\, I \,]) = \mathrm{Row}(A')$. □

This is a partial converse to a previously encountered theorem—that orthogonal row spaces give rise to dual matroids. Hence, for binary matroids $M(A)$, we see that $\mathrm{Row}(A)$ and $\mathrm{Row}(A)^\perp$ are **invariants**: they are determined by the matroid alone and do not depend on the representation.

2.1 Cycles and Cocycles

For a binary matroid $M := M(A)$, define the **cocycle space** to be $\mathscr{C}^*(M) := \mathrm{Row}(A)$, and define the **cycle space** to be $\mathscr{C}(M) := \mathrm{Row}(A)^\perp = \mathscr{C}^*(M)^\perp$. Say that a subset $C \subseteq E(M)$ is a **cycle** if the characteristic vector $\chi_C \in \mathscr{C}(M)$—that is, the columns of $A\big|C$ sum to zero.

Lemma. *$C \subseteq E(M)$ is a cycle iff C is a disjoint union of circuits.*

Proof. Suppose C is a cycle, WLOG nonempty. C is dependent, so it contains some circuit C'. But C' is a minimal dependent set, so its columns must sum to zero—hence both C' and $C - C'$ are cycles. By induction, C is a disjoint union of circuits. The converse is trivial. □

Dually, C^* is a **cocycle** of M if it is a cycle of M^*, or equivalently, the characteristic vector is in the cocycle space $\mathscr{C}^*(M)$. By the Lemma, cocycles are disjoint unions of cocircuits.

Let A be the incidence matrix of a graph G. $C^* \subseteq E(G)$ is a cocycle of $M(G)$ iff it is a cut, and $C \subseteq E(G)$ is a cycle iff $G|C$ is an even subgraph, i.e. every vertex has even degree.

By Lemma 2.3, if C is a cycle and C^* is a cocycle, $|C \cap C^*|$ is even. In fact, C is a cycle iff it has even intersection with every cocycle, and C^* is a cocycle if it has even intersection with every cycle.

Exercise. Let M be any matroid with a circuit C and a cocircuit C^*. Then $|C \cap C^*| \neq 1$.

Lemma. *Let M and N be matroids on a common ground set E. If each circuit of M is a circuit of N, and each cocircuit of M is a cocircuit of N, then $M = N$.*

Proof. Suppose $M \neq N$. Then there exists some circuit C of N that is not a circuit of M. C cannot be dependent in M, or else it would contain an M-circuit; so $C \in \mathcal{I}(M)$. Let $B \supseteq C$ be a basis for M, and take $e \in C$. Then let C^* be the fundamental cocircuit of e in (M, B). C^* is also a cocircuit in N, but this contradicts the exercise above, because $C \cap C^* = \{e\}$. □

29 May **Theorem**. *A matroid $M = (E, \mathcal{I})$ is binary iff, for every circuit C and cocircuit C^*, $|C \cap C^*|$ is even.*

Proof. By Lemma 2.3, one direction is done, so suppose M has $|C \cap C^*|$ even for each circuit C and cocircuit C^*. Let $\mathscr{C}$ be the collection of circuits of M, and $\mathscr{C}^*$ the cocircuits. Then let A be the $\mathscr{C}^* \times E$ matrix where each row C^* is the characteristic vector of that cocircuit, and let A' be the similarly constructed $\mathscr{C} \times E$ matrix for the circuits. By the hypothesis, each row of A is orthogonal with each row of A', so $\mathrm{Row}(A)$ and $\mathrm{Row}(A')$ will be orthogonal row spaces if $\operatorname{rank} A + \operatorname{rank} A' \geq |E|$.

$[\, I \; F \,]$ is a submatrix of A, so $\operatorname{rank} A \geq r(M)$, and likewise $[\, F^{\mathsf{T}} \; I \,]$ is a submatrix of A', so $\operatorname{rank} A' \geq |E| - r(M)$, and thus $M(A)^* = M(A')$, so $M(A)$ is binary. But also every circuit of M is a circuit of $M(A)$, and every cocircuit of M is a circuit in $M(A)^*$, so by the Lemma above, $M = M(A)$ is binary. □

Theorem (Tutte). *A matroid is binary iff it has no $U_{2,4}$-minor.*

Proof. Let M be an excluded minor of the binary matroids. WLOG M is simple. If $r(M) \leq 2$, $M \cong U_{2,4}$, so we may suppose $r(M) \geq 3$. By the previous theorem, there exist a circuit C and a cocircuit C^* with odd intersection. Suppose for a contradiction there exists $e \in C - C^*$. $C - \{e\}$ is a circuit of M/e and C^* remains a cocircuit in M/e. $\big|(C - \{e\}) \cap C^*\big| = |C \cap C^*|$ is odd, so M/e is not binary and M is not an excluded minor, which gives the sought contradiction. By duality, $C = C^*$.

Since $r(M) \geq 3$, $|E - C^*| \geq r(M \backslash C^*) = r(M) - 1 \geq 2$. Let $e, f \in E - C^*$ be distinct. C^* is a cocircuit of M/e, and since M/e is binary, C^* is a cocycle in $M/e \backslash f$. Cocycles partition into cocircuits, so there exists some odd cocircuit $C' \subseteq C^*$. C' is a cocircuit of $M/e\backslash f = M\backslash f/e$ so it is also a cocircuit of $M \backslash f$.

Now, C is a circuit in $M \backslash f$ and C' is an odd cocircuit. But then $C' \subseteq C^* = C$, so $|C \cap C'| = |C'|$ is odd and $M \backslash f$ is not binary, meaning M cannot be an excluded minor if $r(M) \geq 3$. □

2.2 Affine Matroids

1 Jun A binary matroid M is **affine** if $E(M)$ is a cocycle. Observe that, for a graph G, $M(G)$ is affine iff G is bipartite, and $M(G)^*$ is affine iff G is even.

Given $A \in \mathrm{GF}(2)^{r \times E}$, $M(A)$ is affine if there exists $x \in \mathrm{GF}(2)^r$ such that $x^{\mathsf{T}} A = \mathbf{1}^{\mathsf{T}}$ the all-ones row vector, that is, if $\mathbf{1}^{\mathsf{T}}$ is in $\mathrm{Row}(A)$. This can be efficiently checked in GF(2), but the analogue in other fields—even when relaxed to checking whether entries are nonzero—cannot be done efficiently.

Theorem. *A binary matroid is affine iff it has no odd circuits.*

Proof. Suppose M has an odd circuit C. Note that $M\big|C = M(G)$ for G a odd cycle graph. But since G is not bipartite, $M(G) = M\big|C$ is not affine.

Conversely suppose M is not affine. Then there is no solution to $x^\mathsf{T} A = \mathbf{1}^\mathsf{T}$—that is, there must[4] exist some $y \in \mathrm{GF}(2)^E$ such that $Ay = 0$ but $\mathbf{1}^\mathsf{T} y \neq 0$. The support of y is a cycle in M, and since $\mathbf{1}^\mathsf{T} y = 1$, it is a cycle of odd size. Hence it partitions into circuits, at least one of them with odd size. □

Let H be a hyperplane of $\mathrm{PG}(r-1,2)$. Define the **affine geometry** $\mathrm{AG}(r-1,2) := \mathrm{PG}(r-1,2)\backslash H$.[5]

Exercise. Show that a simple rank-r binary matroid is affine iff it is isomorphic to a restriction of $\mathrm{AG}(r-1,2)$.

Observe that a graph G is 4-colourable iff there exists cocycles C and C' of $M(G)$ such that $E(G) = C \cup C'$. Define the **critical number** $\chi(M)$ of a matroid to be the least number of cocycles required to cover $E(M)$. In terms of representations, there exists a representation $M = M(A)$ such that the first $\chi(M)$ rows of A are linearly independent and there are no zero columns. $\chi(M)$ is only well-defined if M has no loops.

Let $\chi(G)$ be the chromatic number of G. Notice $\chi(G) \leq 2^k$ iff $\chi(M(G)) \leq k$. In fact, $\chi(M(G)) = \lceil \log_2 \chi(G) \rceil$.

Let F be a rank-$(r-c)$ flat of $\mathrm{PG}(r-1,2)$. Define the **Bose–Burton geometry** $\mathrm{BB}(r-1,2,c) := \mathrm{PG}(r-1,2)\backslash F$. These generalize the affine geometries, in the sense that $\mathrm{BB}(r-1,2,1) = \mathrm{AG}(r-1,2)$.

Exercise. Let M be a simple rank-r binary matroid. Prove $\chi(M) \leq c$ iff it is a restriction of $\mathrm{BB}(r-1,2,c)$.

Recall the infamous Four-Colour Theorem: every planar graph is 4-colourable. This admits some reformulations in terms of matroids.

- If G is a loopless planar graph, then $E(G)$ is the union of two cocycles of $M(G)$.
- Dually, if G is a bridgeless planar graph, then $E(G)$ is the union of two even subgraphs.

Of course, the above statements are nontrivial, but there exists a "half-dual" form that is much simpler to prove.

Theorem. *If M is a binary matroid, there exists a partition $E(M)$ into a cycle C and a cocycle C^*.*

Proof. Exercise. *Hint*: Suppose $M = M(A)$ and $M^* = M(A')$, and consider whether $\mathbf{1}^\mathsf{T}$ is in $\mathrm{Row}\left(\left[\begin{smallmatrix} A \\ A' \end{smallmatrix}\right]\right)$. □

Corollary. *Every graph has a cut whose removal leaves an even subgraph.* □

Unfortunately, despite its beauty, this result has yet to see a useful application.

3 Extremal Matroid Theory

3 Jun We will now explore some matroid-theoretic analogues of results from extremal graph theory. In general, by mapping edge sets to ground sets, the analogy between graphs and (loopless) matroids is easy to make.

- The analogue of vertex count is, up to an off-by-one error per component, matroid rank.
- Edge-colourings become point-colourings, or equivalently partitions of the ground set.
- Chromatic number is most naturally replaced by critical number—despite that the critical number of a cycle matroid is logarithmic in the chromatic number of the graph.

We will use results from extremal graph theory as our guide, and stick mostly to binary matroids, since the nonbinary generalizations of these results are quite difficult.

3.1 Turán

Recall from graph theory Turán's Theorem:

Theorem (Turán). *If G is a simple graph on n vertices with no m-clique, then $|E(G)| \leq \frac{m-2}{m-1}\binom{n}{2}$.*

[4] "I call it the *Fundamental Theorem of Linear Algebra*, to make students feel bad when they've forgotten it." – Geelen

[5] We retain the 2 parameter, because there is a reasonable—though pernicious—generalization to any finite field.

More accurately, Turán characterized which graphs have the maximal number of edges. Observe that equality can hold when $n = k(m-1)$, in which case G is the complete $(m-1)$-partite graph, each partition of size k.

Let M be a simple binary matroid. A binary matroid is **M-free** if no restriction is isomorphic to M. Let $\mathsf{ex}(M, r)$ denote the maximum number of elements in a simple M-free rank-r binary matroid. Abbreviate $\mathsf{ex}(\mathrm{PG}(t-1, 2), r) =: \mathsf{ex}(t, r)$.

A 3-element circuit in a matroid—isomorphic to $\mathrm{PG}(1, 2) = U_{2,3} = M(K_3)$—is called a **triangle**, and looks like: . Affine matroids have no odd circuits, so $\mathrm{AG}(r-1, 2)$ is triangle-free and

$$\mathsf{ex}(2, r) \geq |\mathrm{AG}(r-1, 2)| = |\mathrm{PG}(r-1, 2)| - |\mathrm{PG}(r-2, 2)| = \frac{2^r - 1}{2-1} - \frac{2^{r-1} - 1}{2-1} = 2^{r-1}.$$

Theorem. $\mathsf{ex}(2, r) = 2^{r-1}$.

Proof. Let M be a simple rank-r triangle-free binary matroid, considered as a restriction of $\mathrm{PG}(r-1, 2)$. Consider some $e \in E(M)$. The number of lines of $\mathrm{PG}(r-1, 2)$ containing e is $\frac{1}{3-1}\big(|\mathrm{PG}(r-1, 2)| - 1\big) = 2^{r-1} - 1$. M can have at most one point on each line, so $|M| \leq 2^{r-1} - 1 + 1 = 2^{r-1}$. □

This argument generalizes easily to the $\mathrm{PG}(t-1, 2)$-free matroids, using Bose–Burton geometries, and obtains the above result as a special case.

Lemma. $\mathrm{BB}(r-1, 2, t-1)$ *is* $\mathrm{PG}(t-1, 2)$*-free.*

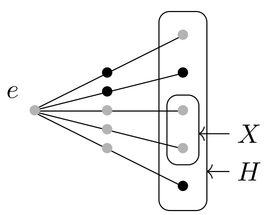
Proof. If F is a rank-t flat in $\mathrm{PG}(r-1, 2)$ and F' is a rank-$(r-(t-1))$ flat, then

$$\sqcap(F, F') = r(F) + r(F') - r(F \cup F') \geq t + (r - t + 1) - r = 1,$$

so $\mathrm{PG}(r-1, 2) \backslash F' = \mathrm{BB}(r-1, 2, t-1)$ is $\mathrm{PG}(t-1, 2)$-free. □

As before, $\mathsf{ex}(t, r) \geq |\mathrm{BB}(r-1, 2, t-1)| = |\mathrm{PG}(r-1, 2)| - |\mathrm{PG}(r-t, 2)| = (2^r - 1) - (2^{r+t-1} - 1) = (1 - 2^{1-t})2^r$.

Theorem (Bose–Burton). *For* $1 \leq t \leq r$, $\mathsf{ex}(t, r) = (1 - 2^{1-t})2^r$.



$M = \{ \bullet \mid \text{grey} \}$

Proof. When $t = 1$, $\mathsf{ex}(1, r) = 0$ so the result holds. Proceed by induction on t. Let M be a simple $\mathrm{PG}(t-1, 2)$-free binary matroid, considered as a restriction of $\mathrm{PG}(r-1, 2)$. Let $e \in E(M)$ and let H be a hyperplane avoiding e. Define $X := \{ f \in E(M) \cap H \mid \{ e, f \} \text{ spans a triangle in } M \}$.

Since M is $\mathrm{PG}(t-1, 2)$-free, $M\big|X$ is $\mathrm{PG}(t-2, 2)$-free. By the induction hypothesis,

$$|X| \leq \mathsf{ex}(t-1, r-1) = (1 - 2^{2-t})2^{r-1}.$$

The number of lines of $\mathrm{PG}(r-1, 2)$ that contain e is $2^{r-1} - 1$, so

$$|M| \leq 1 + (2^{r-1} - 1) + |X| \leq 2^{r-1} + (1 - 2^{2-t})2^{r-1} = \Big(\frac{1}{2} + \frac{1}{2} - 2^{1-t}\Big)2^r = (1 - 2^{1-t})2^r. \quad \square$$

5 Jun It is, of course, no coincidence that we find $\mathsf{ex}(t, r) = |\mathrm{BB}(r-1, 2, t-1)|$.

Theorem (Bose–Burton). *If M is a simple rank-r* $\mathrm{PG}(t-1, 2)$*-free binary matroid and* $|M| = \mathsf{ex}(t, r)$*, then M is isomorphic to* $\mathrm{BB}(r-1, 2, t-1)$.

The proof is left as an exercise; however, the following result will be useful.

Exercise. Let X be a set of points in $\mathrm{PG}(r-1, 2)$ such that no line of $\mathrm{PG}(r-1, 2)$ contains exactly two points of X. Prove that X is a flat.

3.2 Ramsey

Ramsey Theorem. *For each positive integer m there exists sufficiently large n such that in any 2-edge-colouring of K_n, there exists a monochromatic copy of K_m.*

The general form is difficult, so we will beat this result down until it is manageable, and work our way back up.

Bipartite Ramsey Theorem. *For each positive integer m there exists sufficiently large n such that in any 2-edge-colouring of $K_{n,n}$, there exists a monochromatic copy of $K_{m,m}$.*

Density Bipartite Ramsey Theorem. *For each positive integer m and $\varepsilon > 0$, there exists N such that for any $n \geq N$, if a graph G has $|V(G)| = n$ and $|E(G)| \geq \varepsilon\binom{n}{2}$, then G contains a subgraph isomorphic to $K_{m,m}$.*

Since 2-edge-colouring is equivalent to an edge being included or excluded from the edge set of a subgraph, we see that the Density Bipartite Ramsey Theorem implies the Bipartite Ramsey Theorem. One can think of it as a bipartite analogue of Turán's Theorem, though Turán does not itself imply Density Bipartite Ramsey,[6] so we have no reason to expect the matroid analogues to do so. We proceed in finding matroid analogues.

Geometric Density Hales–Jewett Theorem. *For each integer $t \geq 2$ and $\varepsilon > 0$, there exists $R = \mathsf{DHJ}(t, \varepsilon)$, such that for all $r \geq R$, a simple rank-r $\mathrm{AG}(t-1, 2)$–free binary affine matroid M must have $|M| \leq \varepsilon 2^r$.*

Proof. $\mathrm{AG}(1, 2) = U_{2,2}$ so the result is trivial for $t = 2$. Proceed by induction on t. Let R be the maximum of $\mathsf{DHJ}(t-1, \varepsilon^2) + 1$ and the smallest r such that $\varepsilon 2^r - 1 > \varepsilon 2^{r-1}$. Then let $r \geq R$, take M a simple rank-r binary affine matroid, and suppose $|M| > \varepsilon 2^r$. Consider M as a restriction of $\mathrm{PG}(r-1, 2)$, and since M is affine, let H be a hyperplane disjoint from M.

Let $\mathcal{L} := \{\, \mathrm{cl}(A) \mid A \in \binom{M}{2} \,\}$ be the collection of lines in $\mathrm{PG}(r-1, 2)$ that have two elements in M. Note that each line of $\mathcal{L}$ meets H. The average number of lines through a point in H is

$$\frac{|\mathcal{L}|}{|H|} = \frac{\binom{|M|}{2}}{|H|} > \frac{\varepsilon 2^r(\varepsilon 2^r - 1)}{2(2^{r-1} - 1)} > \frac{\varepsilon 2^r(\varepsilon 2^r - 1)}{2^r} = \varepsilon(\varepsilon 2^r - 1) > \varepsilon^2 2^{r-1},$$

so there exists some point $p \in H$ with more than $\varepsilon^2\, 2^{r-1}$ lines of $\mathcal{L}$ through it. Let H' be a hyperplane of $\mathrm{PG}(r-1, 2)$ avoiding p, and let N be the restriction of H' to the collection of points $\{\, q \mid \mathrm{cl}(\{\, p, q \,\}) \in \mathcal{L} \,\}$. Then N is simple, affine, binary, and has rank at most $r-1$; moreover, $|N| > \varepsilon^2\, 2^{r-1}$. Recall that $r-1 \geq \mathsf{DHJ}(t-1, \varepsilon^2)$, so N has an $\mathrm{AG}(t-2, 2)$–restriction, meaning M has an $\mathrm{AG}(t-1, 2)$–restriction. ☐

8 Jun **Geometric Hales–Jewett Theorem**. *For each integer $t \geq 1$, there exists $R = \mathsf{HJ}(t)$ such that for all $r \geq R$, every 2-colouring of $\mathrm{AG}(r-1, 2)$ contains a monochromatic copy of $\mathrm{AG}(t-1, 2)$.*

Proof. Let $r \geq R := \mathsf{DHJ}(t, \frac{1}{4})$, and consider a partition (C, C') of $E(\mathrm{AG}(r-1, 2))$. WLOG take $|C| \geq |C'|$, so $|C| \geq \frac{1}{2}|\mathrm{AG}(r-1, 2)| = \frac{1}{4}2^r$. By the Geometric Density Hales–Jewett Theorem above, $\mathrm{AG}(r-1, 2)\big|C$ contains a restriction isomorphic to $\mathrm{AG}(t-1, 2)$. ☐

Hales–Jewett is in some sense easier than Bipartite Ramsey, because it suffices to simply look at the larger partition. The result also easily extends, by appropriately modifying the ε parameter of $\mathsf{DHJ}(t, \varepsilon)$.

Before proceeding with the Geometric Ramsey Theorem, we present a proof of the graph theoretic analogue.

Theorem (Ramsey). *For all integers $r, b \geq 1$, there exists $N = \mathsf{R}(r, b)$ such that for all $n \geq N$, any 2-edge-colouring of K_n will have a red K_r or a blue K_b.*

Proof. The result is trivial when r or b is 1, so suppose $r, b \geq 2$. Proceed by induction on $r + b$—that is, suppose the induction hypothesis holds for all r', b' such that $r' + b' < r + b$. Let $N := \mathsf{R}(r-1, b) + \mathsf{R}(r, b-1) + 1$. Consider a 2-edge-colouring of some K_n, and pick some vertex $v \in V(K_n)$. Define $R = \{\, w \in V(K_n) \mid vw \text{ is red} \,\}$ and $B = \{\, w \in V(K_n) \mid vw \text{ is blue} \,\}$. Suppose there is neither a red K_r nor a blue K_b. Then $K_n\big|R$ has no red K_{r-1} or blue K_b, and likewise $K_n\big|B$ has no red K_r or blue K_{b-1}. So

$$n = |R| + |B| + 1 < \mathsf{R}(r-1, b) + \mathsf{R}(r, b-1) + 1 = N. \quad \square$$

Geometric Ramsey Theorem. *For all integers $r, b \geq 1$, there exists $N = \mathsf{GR}(r, b)$ such that for all $n \geq N$, any 2-colouring of $\mathrm{PG}(n-1, 2)$ will have a red $\mathrm{PG}(r-1, 2)$ or a blue $\mathrm{PG}(b-1, 2)$.*

Proof. Exercise. *Hint*: Mimic the proof of the Ramsey Theorem: consider a cocircuit C of $\mathrm{PG}(n-1, 2)$, and apply Geometric Hales–Jewett to $\mathrm{PG}(n-1, 2)\big|C$. ☐

[6] "Turán's Theorem does not imply Density Bipartite Ramsey *in this way*—and I say 'in this way' because I have a room full of mathematicians here that will tell me every true thing implies every other true thing." – Geelen

3.3 Erdős–Stone

Let G be a simple graph with chromatic number $t \geq 2$. If H is another graph with $\chi(H) < t$, then H will be G-free. Letting $\mathsf{ex}(G,n)$ be the maximum number of edges in a simple G-free graph on n vertices, it is clear that $\mathsf{ex}(G,n) \geq \frac{t-2}{t-1}\binom{n}{2}$ by Turán's Theorem. In fact, due to Erdős and Stone, we can say something about the asymptotic behaviour as well.

Erdős–Stone Theorem. *If G is a simple graph with $\chi(G) \geq 2$, then $\lim_{n\to\infty} \mathsf{ex}(G,n)/\binom{n}{2} = \frac{\chi(G)-2}{\chi(G)-1}$.*

Remark. We obtain very little information of the asymptotic behaviour of bipartite graphs—if $t = 2$, $\frac{t-2}{t-1} = 0$.

With matroids, we can make a very similar statement by accepting critical number as the proper analogue of chromatic number. Let M be a simple binary matroid with critical number $t \geq 1$. If N is a binary matroid with critical number $\chi(N) < t$, then N must be M-free. Hence $\mathsf{ex}(M,r) \geq |\mathrm{BB}(r-1,2,t-1)| = (1-2^{1-t})\,2^r$, and...

Geometric Erdős–Stone Theorem. *If M is a simple binary matroid, then $\lim_{r\to\infty} \mathsf{ex}(M,r)/2^r = 1 - 2^{1-\chi(M)}$.*

10 Jun We are not quite ready to tackle this theorem. First, an examination of the result for some small cases of t.

Lemma. *If M is a loopless binary matroid, then M has an affine restriction N with $|N| \geq \frac{1}{2}|M|$.*

Proof. Let $M = M(A)$ for $A \in \mathrm{GF}(2)^{r\times E}$. Choose $x \in \mathrm{GF}(2)^r$ uniformly at random, and consider the cocycle C^* of M with characteristic vector $x^{\mathsf{T}}A$. For $e \in E(M)$, $\mathrm{Prob}[\, e \in C^* \,] = \frac{1}{2}$, so by the linearity of expectation, the expected size of C^* is $\frac{1}{2}|M|$. Thus, for some x, the cocycle C^* attains that size or greater, and $M\big|C^*$ is affine. □

Remark. It is an open problem currently, whether such a cocycle/restriction can be found in polynomial time.

Theorem. *For all integers $n \geq 1$ and $\varepsilon > 0$, there exists $R = \mathsf{GES}(n,1,\varepsilon)$ such that, for $r \geq R$,*

$$\mathsf{ex}(\mathrm{AG}(n{-}1,2),r) < \varepsilon 2^r.$$

Proof. Let $r \geq R := \mathsf{DHJ}(n,\frac{\varepsilon}{2})$, and let M be a simple rank-r matroid with $|M| \geq \varepsilon 2^r$. By the Lemma above, M has an affine restriction N such that $|N| \geq \frac{1}{2}|M| \geq \frac{\varepsilon}{2}2^r$. By Geometric Density Hales–Jewett, N has a restriction isomorphic to $\mathrm{AG}(n-1,2)$. □

Theorem. *For all integers $n \geq 2$ and $\varepsilon > 0$, there exists $R = \mathsf{GES}(n,2,\varepsilon)$ such that, for $r \geq R$,*

$$\mathsf{ex}(\mathrm{BB}(n{-}1,2,2),r) < (\tfrac{1}{2}+\varepsilon)\,2^r.$$

12 Jun *Proof* (∗). Let n_1 be $\mathsf{DHJ}(n-1,\frac{\varepsilon}{2})$ and take n_2 to be the least positive integer such that $2^{n_1} \leq \frac{\varepsilon}{2}2^{n_2}$. Then take $r \geq R := \max\{\, n_2, \mathsf{GES}(n_1,1,\frac{1}{2})\,\}$, and let M be a simple binary rank-r matroid with $|M| > (\frac{1}{2}+\varepsilon)2^r$. Consider M as a restriction of $\mathrm{PG}(r-1,2)$, and observe M has a restriction isomorphic to $\mathrm{AG}(n_1-1,2)$.

Hence, there exist flats $F_0 \subseteq F_1$ such that $r(F_0){+}1 = r(F_1) = n_1$ and $F_1{-}F_0 \subseteq E(M)$. Let $M' := M\backslash(F_1{\cap}E(M))$. By the choice of n_2, $|F_1 \cap E(M)| \leq \frac{\varepsilon}{2}2^r$, so $|M'| \geq |M| - \frac{\varepsilon}{2}2^r \geq (\frac{1}{2}+\frac{\varepsilon}{2})2^r$. There exists a flat $F_2 \supseteq F_1$ such that $r(F_2) = r(F_1)+1$ and $|F_2 \cap E(M')| \geq (\frac{1}{2}+\frac{\varepsilon}{2})2^{n_1}$. (*Exercise.*)

Note that $|F_2 - F_1| = 2^{n_1}$. Let $e \in F_1 - F_0$ and let F be a rank-n_1 flat such that $F_0 \subseteq F \subseteq F_2 - \{e\}$. Let $\mathcal{L}$ be the collection of lines containing e and a point in $F - F_0$—$|\mathcal{L}| = |F - F_0| = 2^{n_1-1}$—and $Z \subseteq F - F_0$ be the collection of points f such that the line spanned by e and f is contained in $E(M)$. Observe that

$$(\tfrac{1}{2}+\tfrac{\varepsilon}{2})2^{n_1} \leq |F_2 \cap E(M')| \leq |Z| + |\mathcal{L}| = |Z| + \tfrac{1}{2}2^{n_1},$$

so $|Z| \geq \frac{\varepsilon}{2}2^{n_1}$. Let $N = M\big|Z$. N is an affine matroid of rank at most n_1 with $|N| \geq \frac{\varepsilon}{2}2^r$. Since $n_1 = \mathsf{DHJ}(n{-}1,\frac{\varepsilon}{2})$, N has a restriction $N' \cong \mathrm{AG}(n-2,2)$. Let F' be the flat of M' spanned by N', and let F'' be the rank-n flat spanned by $F' \cup \{e\}$. $F'' - F_0 \subseteq E(M)$ and $r(F'' \cap F_0) = n-2$, so $M\big|(F'' \cap F_0) \cong \mathrm{BB}(m-1,2,2)$. □

The above Theorems imply the Geometric Erdős–Stone Theorem for $t = 1$ and $t = 2$, respectively, since a matroid with, say critical number 1, will be a restriction of $\mathrm{AG}(n-1,2)$. Proving the full result is left as an exercise, though the general inductive step is sketched below.

Theorem. *For all integers $1 \leq t < n$ and $\varepsilon > 0$, there exists $R = \mathsf{GES}(n,t,\varepsilon)$ such that, for $r \geq R$,*

$$\mathsf{ex}(\mathrm{BB}(n-1,2,t),r) < (1 - 2^{1-t} + \varepsilon)\,2^r.$$

Proof (Sketch). Mimic the proof of the result for the $t = 2$, beginning with the crucial difference that $r(F_0)+t-1 = r(F_1) = n_1$. Then, instead of a point e, what is sought is a rank-$(t-1)$ flat F_0^c that is maximally skew to F_0 in F_1. Then extend F_0 to a flat F again maximally skew to F_0^c, now in F_2.

4 Matroid Union and Partition

15 Jun Given matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$, the *matroid partition problem* is to find

$$\max \left\{ |I_1 \cup I_2| \;\middle|\; I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2 \right\}.$$

To tackle this, define the **matroid union** of M_1 and M_2 to be $M_1 \cup M_2 := (E, \{ I_1 \cup I_2 \mid I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2 \})$. The matroid partition problem is thus to find $r(M_1 \cup M_2)$. But we are getting ahead of ourselves.

Theorem. *$M_1 \cup M_2$ is a matroid.*

In order to prove this, we introduce some definitions. Given matroids $M = (E, \mathcal{I})$ and $M' = (E', \mathcal{I}')$, define their **direct sum** $M \oplus M'$ by $E(M \oplus M') := E \sqcup E'$—that is, the disjoint union of the ground sets—and $\mathcal{I}(M \oplus M') := \{ I \sqcup I' \mid I \in \mathcal{I}, I' \in \mathcal{I}' \}$.

Lemma. *$M \oplus M'$ is a matroid. Proof.* Trivial. □

Let F be a flat of the matroid $M = (E, \mathcal{I})$. We construct a new matroid M' by **freely extending** into F—that is, declare $E(M') := E \cup \{\bullet\}$, $M' \backslash \{\bullet\} = M$, and for $X \subseteq E$,

$$r_{M'}(X \cup \{\bullet\}) := \begin{cases} r_M(X) & F \subseteq \mathrm{cl}_M(X), \\ r_M(X) + 1 & \text{otherwise.} \end{cases}$$

Lemma. *M' as constructed above is a matroid. Proof. Exercise.* □

Observe that for any $e \in F$, the new element $\bullet$ obtained by freely extending into F has the property that, for each circuit C containing $\bullet$, $e \in \mathrm{cl}(C)$. Say that $\bullet$ is **freer** than e. In practice, this means that if $I \subseteq E$ is independent and contains e, then $(I - \{e\}) \cup \{\bullet\}$ is independent as well.

Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be matroids on the same ground set. Take a copy $N_1 = (E_1, \mathcal{I}_1')$ of M_1 and a copy $N_2 = (E_2, \mathcal{I}_2')$ of M_2 such that E, E_1, E_2 are all disjoint. Denote the copy of $e \in E$ in N_1 by e_1 and in N_2 by e_2. Construct $M_1 \triangledown M_2$ from $N_1 \oplus N_2$ by freely extending e into the flat $\{ e_1, e_2 \}$, for all $e \in E$. Clearly $M_1 \triangledown M_2$ is a matroid.

Lemma. $(M_1 \triangledown M_2)\big|E = M_1 \cup M_2$.

Proof. Let $I = I_1 \sqcup I_2 \subseteq E$ be an independent set of $M_1 \cup M_2$, WLOG partitioned into disjoint independent subsets $I_1 \in \mathcal{I}_1$ and $I_2 \in \mathcal{I}_2$. Let J_1 and J_2 be the copies of I_1, I_2 in N_1 and N_2, respectively. $J_1 \cup J_2$ is independent in $N_1 \oplus N_2$, and thus also in $M_1 \triangledown M_2$. Each $e \in I$ is freer than either of e_1 or e_2 in $M_1 \triangledown M_2$, so e can replace its copy in $J_1 \cup J_2$ and the set remains independent. Hence I is independent in $M_1 \triangledown M_2$.

Conversely, let $I \subseteq E$ be independent in $M_1 \triangledown M_2$. To show I is $M_1 \cup M_2$ independent, it suffices to find a partition (I_1, I_2) of I such that $\{ e_1 \mid e \in I_1 \} \cup \{ e_2 \mid e \in I_2 \}$ is $(M_1 \triangledown M_2)$–independent. We accomplish this by replacing each element one at a time. So consider an $(M_1 \triangledown M_2)$–independent set I such that $e \in I \cap E$. $e \in \mathrm{cl}(\{ e_1, e_2 \})$, so $I - \{e\}$ cannot span both e_1 and e_2. Hence, at least one of $(I - \{e\}) \cup \{e_1\}$ or $(I - \{e\}) \cup \{e_2\}$ is independent. By induction, $(M_1 \triangledown M_2)$–independent subsets of E are $(M_1 \cup M_2)$–independent. □

Proof of Theorem. By the Lemmas above, $M_1 \triangledown M_2$ is a matroid and $M_1 \cup M_2$ is a restriction of $M_1 \triangledown M_2$. □

Having constructed and grasped the nature of $M_1 \cup M_2$, we seek to answer the matroid partition problem by determining the rank of $M_1 \cup M_2$.

Lemma. *For any $A \subseteq E$, $r(M_1 \cup M_2) \le |E - A| + r_{M_1}(A) + r_{M_2}(A)$.*

Proof. $(M_1 \cup M_2)\big|A = \big(M_1\big|A\big) \cup \big(M_2\big|A\big)$, so

$$r(M_1 \cup M_2) \le |E - A| + r\big((M_1 \cup M_2)\big|A\big) \le |E - A| + r\big(M_1\big|A\big) + r\big(M_2\big|A\big). \quad \square$$

17 Jun There is a min-max formula hiding in this bound, as we shall see—$r(M_1 \cup M_2)$ has no reason to be any lower.

Lemma. *If $M_1 \cup M_2$ has no coloops, then $r(M_1 \cup M_2) = r(M_1) + r(M_2)$.*

Proof. Consider the construction of $M_1 \triangledown M_2$. For $S \subseteq E$ define $\tilde{S} = (E - S) \cup \{\, e_1, e_2 \mid e \in S \,\}$. So $\tilde{E} = E_1 \cup E_2$, meaning $r(\tilde{E}) = r(M_1) + r(M_2)$. Moreover, $\tilde{\varnothing} = E$, so the goal is to prove $r(\tilde{\varnothing}) = r(\tilde{E})$. Choose minimal $S \subseteq E$ such that $r(\tilde{S}) = r(\tilde{E})$, and suppose for a contradiction there exists $e \in S$. Then define $T := S - \{e\}$, and observe $r(\tilde{T}) < r(\tilde{S}) = r(\tilde{E})$ by minimality of S, and $\tilde{T} = (\tilde{S} - \{\, e_1, e_2 \,\}) \cup \{e\}$.

Either e is a coloop of $(M_1 \triangledown M_2)\big|\tilde{T}$ or it is not. If it is, then T is spanned by $\{\, f_1, f_2 \mid f \in T \,\} \subseteq \tilde{T} - \{e\}$, so e is also a coloop of $(M_1 \triangledown M_2)\big|(T \cup \tilde{T})$. But $E \subseteq T \cup \tilde{T}$, so e would then be a coloop of $(M_1 \triangledown M_2)\big|E = M_1 \cup M_2$ as well, contradicting the hypothesis. If it is not, on the other hand, then e is spanned by $\tilde{T} - \{e\} = \tilde{S} - \{\, e_1, e_2 \,\}$. e is freer than either of e_1 or e_2, so both of those vertices are also spanned by $\tilde{S} - \{\, e_1, e_2 \,\}$, and then

$$r(\tilde{T}) = r(\tilde{T} - \{e\}) = r(\tilde{S} - \{\, e_1, e_2 \,\}) = r(\tilde{S}) = r(\tilde{E}),$$

which contradicts the minimality of S. Thus, there does not exist an $e \in S$. □

Theorem. *If X is the set of coloops of $M_1 \cup M_2$, then $r(M_1 \cup M_2) = |X| + r_{M_1}(E - X) + r_{M_2}(E - X)$.*

Proof. $(M_1 \cup M_2)\backslash X$ has no coloops, and $r\big((M_1 \cup M_2)\backslash X\big) = r(M_1 \cup M_2) - |X|$. Moreover, $(M_1 \cup M_2)\backslash X$ is equal to the matroid union $(M_1\backslash X) \cup (M_2\backslash X)$, so by the Lemma,

$$\begin{aligned} r(M_1 \cup M_2) &= |X| + r\big((M_1 \cup M_2)\backslash X\big) = |X| + r\big((M_1\backslash X) \cup (M_2\backslash X)\big) = |X| + r(M_1\backslash X) + r(M_2\backslash X) \\ &= |X| + r_{M_1}(E - X) + r_{M_2}(E - X). \quad \square \end{aligned}$$

Corollary (Matroid 2-Partition Theorem). $r(M_1 \cup M_2) = \min\limits_{A \subseteq E} \big(\, |E - A| + r_{M_1}(A) + r_{M_2}(A) \,\big)$. □

Matroid Partition Theorem (Edmonds). *If $M_1, ..., M_k$ are matroids on a common ground set, then*

$$r(M_1 \cup \cdots \cup M_k) = \min_{A \subseteq E} \big(\, |E - A| + r_{M_1}(A) + \cdots + r_{M_k}(A) \,\big).$$

The proof is left as an exercise. Observe that this proof can be made constructive: there is a canonical choice of subset A for which the rank is attained.

In proving the above result, Edmonds also gave an algorithm[7] for computing $r(M_1 \cup \cdots \cup M_k)$ in $O(k^3|E|^3)$, given the matroids $M_1, ..., M_k$. For the purposes of matroid algorithms, we say that a matroid M is **given** by its ground set $E(M)$ and access to some **rank oracle** that computes $r_M(X)$ for any $X \subseteq E(M)$ in 1 time unit.

4.1 Packing Bases and Independent Covers

Given a matroid M and positive integer k, the *basis packing problem* is to determine whether M has k disjoint bases. Observe that this can be reduced to the matroid partition problem: M has k disjoint bases iff $r(\underbrace{M \cup \cdots \cup M}_{k \text{ times}}) = kr(M)$.[8]

If B is a basis for the matroid M, and $e \in E(M)$, then B contains a basis of M/e. If M has k disjoint bases, applying this argument to each basis shows that, for any $X \subseteq E(M)$, M/X has k disjoint bases as well. Further, observe trivially that to have k disjoint bases, necessarily $|M| \geq kr(M)$. In some sense, these two observations together give a sufficient condition:

Theorem (Tutte, Nash-Williams). *A matroid M has k disjoint bases iff $|M/X| \geq kr(M/X)$ for all $X \subseteq E(M)$.*

19 Jun *Proof.* One direction is obvious, so suppose M does not have k disjoint bases. Then $r(M \cup \cdots \cup M) < kr(M)$. By the Matroid Partition Theorem, there exists $X \subseteq E$ such that $r(M \cup \cdots \cup M) = |E - X| + kr_M(X)$. Thus, $|M/X| = |E - X| = r(M \cup \cdots \cup M) - kr_M(X) < kr(M) - kr_M(X) = kr(M/X)$. □

[7] Covered in CO 450, Combinatorial Optimization.

[8] *Geelen*: [...] if your $M \cup \cdots \cup M$ has rank $k\,r(M)$. *Class*: (begins giggling)

Corollary. *If a graph G is $2k$-edge-connected, then G has k disjoint spanning trees.*

Proof. G is $2k$-edge-connected, so $H := G/X$ is as well for any $X \subseteq E(G)$.

$$|E(H)| = \frac{1}{2} \sum_{v \in V(H)} \deg(v) \geq \frac{1}{2} \sum_{v \in V(H)} 2k = k|V(H)| \geq k(|V(H)| - 1) = kr(M(H)) = kr(M(G)/X).$$

By the previous Theorem, the result follows. $\square$

Given a matroid M and positive integer k, the *independent covering problem* is to determine whether $E(M)$ can be covered by k independent sets. An immediate application would be to see if a graph can be covered by k forests. Observe that this also reduces to the matroid partition problem: M can be covered with k independent sets iff $r(\underbrace{M \cup \cdots \cup M}_{k \text{ times}}) = |E(M)|$.

Just as having k disjoint bases was closed under contraction, being coverable by k independent sets is closed under deletion: if we can cover M, easily we can cover any restriction of M. Observe also that if we can cover M with k independent sets, M can't have too many elements: $|M| \leq kr(M)$.

Theorem. *A matroid M can be covered by k independent sets iff $|X| \leq kr_M(X)$ for all $X \subseteq E(M)$.*

Proof. One direction is obvious, so suppose M can't be covered by k independent sets—$r(M \cup \cdots \cup M) < |E(M)|$. By the Matroid Partition Theorem, there exists $X \subseteq E$ such that $r(M \cup \cdots \cup M) = |E(M) - X| + kr_M(X)$, so $|X| = |E(M)| - |E(M) - X| = |E(M)| - r(M \cup \cdots \cup M) + kr_M(X) > kr_M(X)$. $\square$

Corollary ("Eight-colour Theorem"). *Every simple planar graph can be covered by three forests.*

Proof. By the result of the previous theorem, it suffices to prove that, for all simple connected planar graphs G, $|E(G)| \leq 3r(M(G)) = 3(|V(G)| - 1)$. We may assume $|V(G)| \geq 3$, so by planarity $|E(G)| \leq 3|V(G)| - 6$. $\square$

Corollary. *Let $\mathscr{M}$ be a class of binary matroids closed under restriction, and let k be a positive integer such that for every simple $M \in \mathscr{M}$, $|M| \leq kr(M)$. Then for every loopless $M \in \mathscr{M}$, $\chi(M) \leq k$.*

Proof. Let $M \in \mathscr{M}$ be loopless and let $\overline{M}$ be its simplification. Observe $\chi(M) = \chi(\overline{M})$. By the Theorem, there is a partition $(I_1, ..., I_k)$ of $|E(\overline{M})|$ into independent sets. $\overline{M}|I_n$ has no odd circuits, so it is affine, and thus $\overline{M}$ has cocycles $C_n \supseteq I_n$, for each $1 \leq n \leq k$. $C_1 \cup \cdots \cup C_n = E(\overline{M})$, so $\chi(\overline{M}) \leq k$. $\square$

Exercise (Jaeger). If M is a cographic matroid without coloops, show that $\chi(M) \leq 3$.

Equivalent to the above exercise is the result that every bridgeless graph is the union of three even subgraphs.

Exercise. If G is a 4-edge-connected graph, then G is the union of two even subgraphs.

This implies that planar graphs with girth four are 4-colourable.

4.2 Matroid Intersection

22 Jun Let M_1 and M_2 be matroid on a common ground set E. While the union of M_1 and M_2 is a matroid, their intersection is not always a matroid. Let $\nu(M_1, M_2)$ denote the maximum size of a mutually independent set.

Lemma. $\nu(M_1, M_2) = r(M_1 \cup M_2^*) - r(M_2^*)$.

Proof. Let $\mathscr{B}_1$ be the collection of bases of M_1 and $\mathscr{B}_2$ be the collection of bases of M_2. For $B_1 \in \mathscr{B}_1$ and $B_2 \in \mathscr{B}_2$, $|B_1 \cap B_2| + r(M_2^*) = |B_1 \cap B_2| + |E - B_2| = |(B_1 \cap B_2) \cup (E - B_2)| = |B_1 \cup (E - B_2)|$. Hence,

$$\nu(M_1, M_2) + r(M_2^*) = \max_{B_1, B_2} |B_1 \cap B_2| + r(M_2^*) = \max_{B_1, B_2} |B_1 \cup (E - B_2)| = r(M_1 \cup M_2^*). \quad \square$$

Matroid Intersection Theorem (Edmonds). $\nu(M_1, M_2) = \min_{A \subseteq E} \big(r_{M_1}(A) + r_{M_2}(E - A) \big)$.

Proof.
$$\begin{aligned}\nu(M_1, M_2) &= r(M_1 \cup M_2^*) - r(M_2^*) = \min_{A \subseteq E} \big(|E - A| + r_{M_1}(A) + r_{M_2^*}(A) \big) - r(M_2^*) \\ &= \min_{A \subseteq E} \big(|E| - |A| + r_{M_1}(A) + (|A| - r(M_2) + r_{M_2}(E - A)) - (|E| + r(M_2)) \big) \\ &= \min_{A \subseteq E} \big(r_{M_1}(A) + r_{M_2}(E - A) \big). \quad \square\end{aligned}$$

An easy application of this is to the problem of bipartite matchings. Given a bipartite graph $G = (V, E)$ with bipartition (A, B), define a matroid $M_A = (E, \mathcal{I}_A)$ by saying $I \in \mathcal{I}(M_A)$ iff at most one edge in I is incident at each $a \in A$. Observe that M_A is a matroid by the following representation: $M_A = M(T)$ for $T \in \mathrm{GF}(2)^{A \times E}$, where $T_{ae} = 1$ iff e is incident to a. Define the matroid M_B similarly, and then see that $\nu(G) = \nu(M_A, M_B)$.

Recall that a **vertex cover** of a graph G is a set $S \subseteq V(G)$ of vertices such that $E(G - S) = \varnothing$. Given the Matroid Intersection Theorem as we have it above, König's Theorem falls right out.

König's Theorem. *The maximum size of a matching in a bipartite graph is equal to the minimum size of a vertex cover.*

Proof. Let $G = (V, E)$ be a bipartite graph, with bipartition (A, B). Construct M_A and M_B as above. By the Matroid Intersection Theorem, there is a partition (X_A, X_B) of E such that $\nu(M_A, M_B) = r_A(X_A) + r_B(X_B)$. Let C_A be the vertices of A incident at edges of X_A and likewise C_B the vertices of B incident at vertices of X_B. Then $C := C_A \cup C_B$ is a cover and $|C| = |C_A| + |C_B| = r_A(X_A) + r_B(X_B) = \nu(M_A, M_B) = \nu(G)$. $\square$

In fact, we can do better. We proved the Matroid Partition Theorem constructively, so we obtain Matroid Intersection constructively as well. This can be carried into König's Theorem. Compare the results of §1.5.

Theorem (Mendelsohn). *Let G be a bipartite graph with bipartition (A, B). Let X be the avoidable vertices in A and let $C := (A - X) \cup N(X)$. Then C is a cover and $|C| = \nu(G)$. Proof. Exercise.* $\square$

5 Matroid Connectivity

24 Jun Recall local connectivity: if M is a matroid and $X, Y \subseteq E(M)$, then $\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y)$.

Proposition. *If $X' \subseteq X$, then $\sqcap(X', Y) \le \sqcap(X, Y)$.*

Proof. By submodularity, $r(X) + r(X' \cup Y) \ge r(X') + r(X \cup Y)$, so

$$\sqcap(X, Y) = r(X) + r(Y) - r(X \cup Y) \ge r(X') + r(Y) - r(X' \cup Y) = \sqcap(X', Y). \quad \square$$

Exercise. Show that if $r(X') = r(X)$, $\sqcap(X', Y) = \sqcap(X, Y)$.

Lemma. *Let M be a matroid and let (X, Y) partition $E(M)$. Then $M = M|X \oplus M|Y$ iff $r(X) + r(Y) = r(M)$.*

Proof. If $M = M|X \oplus M|Y$ then $r(X) + r(Y) = r(X \cup Y) = r(M)$. Conversely, suppose $r(X) + r(Y) = r(M)$. Then, if $X' \subseteq X$ and $Y \subseteq Y'$, the Proposition implies that $\sqcap(X', Y') \le \sqcap(X, Y) = r(X) + r(Y) - r(M) = 0$, so $r(X') + r(Y') = r(X' \cup Y')$, meaning $M = M|X \oplus M|Y$. $\square$

Say that a matroid $M = (E, \mathcal{I})$ is **connected** if, for each proper partition (X, Y) of E, $r(X) + r(Y) > r(M)$—or equivalently, $\sqcap(X, Y) \ge 1$. Define a **connectivity function** $\lambda_M : 2^E \to \mathbb{Z}$ by

$$\lambda(X) := \sqcap(X, E - X) = r(X) + r(E - X) - r(M).$$

Because of the properties of rank and local connectivity, this function has a number of useful properties.

Lemma. $\lambda_M(X) = r_M(X) + r_{M^*}(X) - |X|$.

Proof. $r(X) + r^*(X) = r(X) + |X| - r(M) + r(E - X) = \lambda(X) + |X|$. $\square$

Lemma. *λ_M is submodular and invariant under complements and duality—$\lambda_M(X) = \lambda_M(E - X) = \lambda_{M^*}(X)$.* $\square$

Lemma. *If X and C are disjoint, $\lambda_{M/C}(X) = \lambda_M(X) - \sqcap_M(X, C)$.*

Proof. $\lambda_{M/C}(X) = r_{M/C}(X) + r_{M^* \backslash C}(X) - |X| = r_M(X) - \sqcap_M(X, C) + r_{M^*}(X) - |X| = \lambda_M(X) - \sqcap_M(X, C)$. $\square$

Lemma (Bixby–Coullard Inequality). *Let M be a matroid and $e \in E(M)$. If (C, C') and (D, D') are partitions of $E - \{e\}$, then $\lambda_{M/e}(C) + \lambda_{M\backslash e}(D) \geq \lambda_M(C \cap D) + \lambda_M(C' \cap D') - 1$.*

Proof. Bashing out definitions and submodularity:

$$\begin{aligned}
&\lambda_{M/e}(C) + \lambda_{M\backslash e}(D) \\
&= r_{M/e}(C) + r_{M/e}(C') - r(M/e) + r_{M\backslash e}(D) + r_{M\backslash e}(D') - r(M\backslash e) \\
&= r_M(C \cup \{e\}) + r_M(C' \cup \{e\}) - r(M) - r_M(\{e\}) + r_M(D) + r_M(D') - r(M\backslash e) \\
&\geq r_M(C \cap D) + r_M(C \cup D \cup \{e\}) + r(C' \cap D') + r(C' \cup D' \cup \{e\}) - \sqcap_M(\{e\}, E - \{e\}) - 2r(M) \\
&= \lambda_M(C \cap D) + \lambda_M(C' \cap D') - \lambda_M(\{e\}) \geq \lambda_M(C \cap D) + \lambda_M(C' \cap D') - 1. \quad \square
\end{aligned}$$

Despite looking hefty, Bixby–Coullard is actually very useful for connectivity. One application is as follows.

Theorem. *If M is a connected matroid and $e \in E(M)$, then at least one of M/e and $M\backslash e$ is connected.*

Proof. Suppose for a contradiction that neither is connected. Then there exist proper partitions (C, C') and (D, D') of $E(M) - \{e\}$ such that $\lambda_{M/e}(C) = 0 = \lambda_{M\backslash e}(D)$. We may assume that $C \cap D$ and $C' \cap D'$ are nonempty. By the Bixby–Coullard inequality,

$$0 = \lambda_{M/e}(C) + \lambda_{M\backslash e}(D) \geq \lambda_M(C \cap D) + \lambda_M(C' \cap D') - 1,$$

and since λ_M is nonnegative integer–valued, either $\lambda_M(C \cap D) = 0$ or $\lambda_M(C' \cap D') = 0$. But each possibility would contradict the connectivity of M, so M/e and $M\backslash e$ cannot both be disconnected. $\square$

Exercise. A matroid M is connected iff for every disjoint pair $e, f \in E(M)$, there exists a circuit $C \supseteq \{ e, f \}$.

Corollary. *Let G be a graph. $M(G)$ is connected iff G is loopless and 2-connected.*

Proof. By Menger's Theorem from graph theory, G is loopless and 2-connected if each pair of distinct edges is contained in a circuit. The result follows by the previous exercise. $\square$

26 Jun Let B be the basis of a matroid $M = (V, \mathcal{I})$. Define the **fundamental graph** of (M, B) by $\mathrm{FG}(M, B) := (V, E)$, where $E = \{ uv \in B \times (E - B) \mid (B - \{u\}) \cup \{v\} \text{ is a basis} \}$. Equivalently, if $F = F(M, B)$, the adjacency matrix of $\mathrm{FG}(M, B)$ is $\left[\begin{smallmatrix} 0 & F \\ F^{\mathsf{T}} & 0 \end{smallmatrix}\right]$.

Exercise. Prove that M is connected iff $\mathrm{FG}(M, B)$ is connected.

Let $M = (E, \mathcal{I})$ be a matroid, and let $S, T \subseteq E$ be disjoint. The **connectivity** of S and T is

$$\kappa_M(S, T) := \min \{ \lambda_M(X) \mid S \subseteq X \subseteq E - T \}.$$

Observe that $\kappa_M(S, T) = \kappa_M(T, S) = \kappa_{M^*}(S, T)$.

Lemma. *If N is a minor of M with $S, T \subseteq E(N)$ disjoint, then $\kappa_N(S, T) \leq \kappa_M(S, T)$.*

Proof. It suffices to consider the case where $N = M\backslash e$, by induction and duality. Let $X \subseteq E(M)$ be such that $S \subseteq X \subseteq E - T$ and $\lambda_M(X) = \kappa_M(S, T)$. Without loss of generality, we may assume $e \notin X$. Then $\kappa_M(S, T) = \lambda_M(X) \geq \lambda_N(X) \geq \kappa_N(S, T)$. $\square$

Tutte's Linking Theorem. *Let M be a matroid with $S, T \subseteq E(M)$ disjoint. Then there exists a minor N of M such that $E(N) = S \cup T$ and $\lambda_N(S) = \kappa_M(S, T)$.*

Proof. For brevity, let $k = \kappa_M(S, T)$. The result is trivial when $E(M) = S \cup T$, so it suffices to show that for $e \in E(M) - S \cup T$, either $\kappa_{M/e}(S, T) = k$ or $\kappa_{M\backslash e}(S, T) = k$. Suppose for a contradiction that both are strictly less than k. Then there exist partitions (C, C') and (D, D') of $E(M) - \{e\}$ such that $S \supseteq C \cap D$, $T \supseteq C' \cap D'$, $\lambda_{M/e}(C) \leq k - 1$ and $\lambda_{M\backslash e}(D) \leq k - 1$. By the Bixby–Coullard Inequality,

$$2k - 1 \geq \lambda_{M/e}(C) + \lambda_{M\backslash e}(D) + 1 \geq \lambda_M(C \cap D) + \lambda_M(C' \cap D').$$

It must be that either $\lambda_M(C \cap D) \leq k - 1$ or $\lambda_M(C' \cap D') \leq k - 1$—but that would mean $\kappa_M(S, T) \leq k - 1$. $\square$

Exercise. Prove that $\kappa_M(S, T) = \nu(M\backslash S/T, M/S\backslash T) - r(M) + r(S) + r(T)$, and hence derive Tutte's Linking Theorem from the Matroid Intersection Theorem.

As a consequence of the previous result, κ_M can be computed efficiently.

Exercise (∗). Derive the Matroid Intersection Theorem from Tutte's Linking Theorem.

5.1 Application: Menger's Theorem

Let $G = (V, E)$ be a graph. A **separation** of G is a pair of edge-disjoint subgraphs (H, H') such that $H \cup H' = G$. The order of the separation is $\operatorname{ord}(H, H') = |V(H) \cap V(H')|$.

Lemma. *For a separation (H, H') of G, $\lambda_{M(G)}(E(H)) = \operatorname{ord}(H, H') - \operatorname{comps}(H) - \operatorname{comps}(H') + \operatorname{comps}(G)$.* □

Notably, when all of G, H, H' are connected, $\lambda_{M(G)}(E(H)) = \operatorname{ord}(H, H') - 1$. In some sense, this is the only interesting case—if G is disconnected, there is an order-0 separation.

29 Jun For $S, T \subseteq V(G)$, let $\kappa_G(S, T)$ be the minimum order of a separation (H_S, H_T) of G such that $S \subseteq V(H_S)$ and $T \subseteq V(H_T)$.

Lemma. *Let H_S, H_T be edge-disjoint subgraphs of G. Then $\kappa_G(V(H_S), V(H_T)) = \kappa_{M(G)}(E(H_S), E(H_T)) + 1$ if G, H_S, and H_T are all connected.*

Proof. Easily, $\kappa_G(V(H_S), V(H_T)) \leq \kappa_{M(G)}(E(H_S), E(H_T)) + 1$, since $\lambda(E(H)) \leq \operatorname{ord}(H, H') - 1$ for every separation (H, H') of G. So let $X \subseteq E$ such that $E(H_S) \subseteq X \subseteq E - E(H_T)$ and $\lambda(X) = \kappa_{M(G)}(E(H_S), E(H_T))$. Let $G_1 := G\big|X$ and $G_2 := G\big|(E - X)$. We may assume X minimizes $\operatorname{comps}(G_1) + \operatorname{comps}(G_2)$, and if both graphs are connected, we are done. Up to symmetry, suppose G_1 is not connected.

Let H be some component of G_1 not containing H_S, and let H' be the remainder. Observe that $\operatorname{comps}(G_2 \cup H) \geq \operatorname{comps}(G_2) - (\operatorname{ord}(H, G_2) - 1)$, and then

$$\begin{aligned}
\lambda(E(H')) &= \operatorname{ord}(H', G_2 \cup H) - \operatorname{comps}(H') - \operatorname{comps}(G_2 \cup H) + 1 \\
&\leq \operatorname{ord}(H', G_2 \cup H) - \operatorname{comps}(H') - \operatorname{comps}(G_2) + \operatorname{ord}(H, G_2) \\
&= \lambda(E(G_1)) = \lambda(X).
\end{aligned}$$

However, $\operatorname{comps}(G_1) + \operatorname{comps}(G_2) > \operatorname{comps}(H') + \operatorname{comps}(G_2 \cup H)$, so G_1 being disconnected is contradictory. □

Theorem (Menger). *The maximum number of vertex-disjoint (S, T)-paths in G is equal to $\kappa_G(S, T)$.*

Proof. Let k be the maximum number of vertex-disjoint (S, T)-paths. Clearly, $k \leq \kappa_G(S, T)$. Adding edges into G with both endpoints in S or T does not change either k or $\kappa_G(S, T)$, so WLOG letting H_S and H_T be edge-disjoint connected subgraphs of G such that $V(H_S) = S$ and $V(H_T) = T$. $\kappa_{M(G)}(E(H_S), E(H_T)) = \kappa_G(S, T) - 1$ by the previous Lemma.

By Tutte's Linking Theorem, there exists a minor N of $M(G)$ on the ground set $E(N) = E(H_S) \cup E(H_T)$ and $\lambda_N(E(H_S)) = \kappa_{M(G)}(E(H_S), E(H_T))$. Suppose $N = M/C\backslash D$. Then $\sqcap_{M(G)}(E(H_S), E(H_T)) = \kappa_G(S, T) - 1$. Let $H = G\big|C$. By the above, there are $\kappa_G(S, T)$ components of H containing a vertex of both S and T. Hence, $k \geq \kappa_G(S, T)$. □

5.2 Tutte Connectivity

A partition (X, Y) of the ground set of a matroid M is a **k-separation** of M if $\lambda_M(X) < k \leq |X|, |Y|$. The lower bound on the sizes of X and Y is a nondegeneracy condition, because $\lambda(X) \leq \min\{\,|X|, |Y|\,\}$. If (X, Y) is a 1-separation, then X and Y are nonempty and skew; if it is a 2-separation, then X and Y have at least two elements each and their local connectivity is at least 1. Say that a matroid is (**Tutte**) **k-connected** if it has no k'-separation for $k' < k$.

2-connectedness means the matroid is connected, by definition. Notice that M is k-connected iff M^* is. However, notice that if M contains a triangle T and $|M| \geq 6$, then $\lambda(T) = r(T) + r^*(T) - |X| \leq r(T) = 2$, so $(T, E(M) - T)$ is a 3-separation. Hence, matroids like $M(K_n)$ and $\mathrm{PG}(n-1, 2)$ are not 4-connected when $n \geq 4$.

Exercise (∗). Show that for each k, there exists a graph G such that $M(G)$ is k-connected and $E(G) \geq 2k$.

3 Jul A partition (X, Y) of $E(M)$ is a **vertical k-separation**[9] of M if $\lambda(X) < k \le r(X), r(Y)$. M is **vertically k-connected** if it has no vertical k'-separation for $k' < k$. Observe that vertical connectivity is not preserved under duality—consider for instance $U_{4,6}$.

Theorem (Cunningham). *Let G be a connected graph and $k \ge 1$. Then G is k-connected iff $M(G)$ is vertically k-connected.*

Proof. The result is trivial when $k = 1$. Proceeding by induction, suppose G is $(k-1)$-connected and $M(G)$ is vertically $(k-1)$-connected. If G is not k-connected, then G has a separation (H, H') of order less than k, and neither $V(H) - V(H')$ or $V(H') - V(H)$ is empty. G is $(k-1)$-connected, so it is connected and hence H and H' are as well. Thus, $\lambda(E(H)) = (k-1) - 1 = k-2$. $|V(H)|, |V(H')| \ge k$, so $r(E(H)), r(E(H')) \ge k-1$, meaning $(E(H), E(H'))$ is a vertical $(k-1)$-separation, and $M(G)$ is not vertically k-connected.

Conversely, suppose $M(G)$ is not vertically k-connected but G is k-connected. Take a vertical k-separation (A, B) of $M(G)$. Neither $G|A$ nor $G|B$ is connected, so there exist $u, v \in V(G)$ which are in distinct components of both $G|A$ and $G|B$. (*Exercise.*) Since G is k-connected, there exist k internally vertex-disjoint uv-paths. Each path must contain an edge of A and an edge of B. Contract one path to identify u with v, and for the remaining $k-1$ paths, contract all but an A-edge and a B-edge. Deleting everything else, leaving only the minor N on the ground set $\{ a_i, b_i \mid 1 \le i \le k-1 \}$, we find a contradiction:

$$k - 2 \ge \lambda_{M(G)}(A) \ge \kappa_{M(G)}(\{ a_1, ..., a_{k-1} \}, \{ b_1, ..., b_{k-1} \}) \ge \sqcap_N(\{ a_1, ..., a_{k-1} \}, \{ b_1, ..., b_{k-1} \}) = k - 1. \quad \square$$

Exercise. Let M be a matroid with $|M| \ge 2k$. Show that M is Tutte k-connected iff it is vertically k-connected and has **girth**—length of shortest circuit—at least k.

Theorem. *For a connected graph G, $M(G)$ is k-connected iff G is k-connected and has girth at least k.* $\square$

6 Jul We now investigate the case of $k = 3$. Given a graph G, $M(G)$ is 3-connected iff G is 3-connected and has girth at least 3—in other words, G is simple. Say that a matroid M is **internally 3-connected** if M is connected—that is, 2-connected—and for each 2-separation (X, Y) of M, either $|X| = 2$ or $|Y| = 2$. Let M be a 3-connected matroid, unless otherwise specified.

If M is internally 3-connected and (X, Y) is a 2-separation with $|X| = 2$, then X is a parallel pair or a series pair, since $1 = \lambda(X) = r(X) + r^*(X) - 2$. Thus, if $|M| \ge 4$, $e \in E(M)$, and M/e is internally 3-connected, then since M/e has no series pairs, so the simplification of M/e is 3-connected. Dually, if $M \backslash e$ is internally 3-connected, it has no parallel pairs, so its cosimplification[10] is 3-connected.

Bixby's Lemma. *Let $e \in E(M)$. Then one of M/e or $M \backslash e$ is internally 3-connected.*

Proof. Suppose otherwise. Then there exist 2-separations (C, C') and (D, D') in M/e and $M \backslash e$, respectively, such that $|C|, |C'|, |D|, |D'| \ge 3$. WLOG, assume $|C \cap D|, |C' \cap D'| \ge 2$. By the Bixby–Coullard Inequality, $3 \ge \lambda_{M/e}(C) + \lambda_{M \backslash e}(D) + 1 \ge \lambda_M(C \cap D) + \lambda_M(C' \cap D')$, so either $\lambda_M(C \cap D)$ or $\lambda_M(C' \cap D')$ is at most 1, contradicting the 3-connectivity of M. $\square$

~~**Theorem**~~ *Exercise.* Let e be an element of some matroid M. Then for N equal to either M/e or $M \backslash e$, and any partition (X, Y) of $E(N)$, there is a partition (A, B) of either X or Y with $\lambda_M(A), \lambda_M(B) \le \lambda_N(X)$.

Corollary. *If $|M| \ge 4$, then M has a $U_{2,4}$-minor or a $M(K_4)$-minor. Proof. Exercise.* $\square$

Knowing that every element of a 3-connected matroid gives something internally 3-connected when minored out, it behooves us to determine if there exist elements which can be minored out to give a 3-connected matroid.

If M/e is internally 3-connected but not 3-connected, then it must have been that e belonged to a triangle. Likewise, if $M \backslash e$ is only internally 3-connected, then e must be an element of a **triad**—a three-element cocircuit, the dual of a triangle.

Lemma. *Let $e \in E(M)$. If e is in neither a triangle nor a triad, then one of M/e or $M \backslash e$ is 3-connected.* $\square$

At this point, up to duality, we may assume that M has a triangle.

[9] *vertical* (n) 2. of, relating to, or situated at a vertex or vertices. (Despite sounding more natural, **verticial* is not a real word.)
[10] Simplification under duality: remove coloops and collapse coparallel classes.

Lemma. *Let e, f, g, h be distinct elements of M. If $\{ e, f, g \}$ is a triangle and $\{ f, g, h \}$ is a triad, then $M \backslash e$ is 3-connected.*

Proof. Left as an exercise. *Hint*: Consider M/e.

8 Jul **Tutte's Triangle Lemma**. *Let $T = \{ e, f, g \}$ be a triangle of M and $|M| \geq 4$. If e is not contained in a triad, then at least one of $M \backslash e, M \backslash f, M \backslash g$ is 3-connected.*

Proof. Suppose none of $M\backslash e, M\backslash f, M\backslash g$ are 3-connected. Then some small cases can be eliminated, and we may assume $|M| \geq 8$. (*Exercise. Hint*: Recall that if $|M| \leq 7$, then $r(M) \leq 3$ or $r(M^*) \leq 3$.) Consider a 2-separation (A, B) of $M\backslash e$ with $f \in A$. If $g \in A$ as well, then, since e is spanned by f and g,

$$\lambda_M(B) = r_M(A \cup \{e\}) + r_M(B) - r(M) = r_M(A) + r_M(B) - r(M\backslash e) = \lambda_{M\backslash e}(A) = 1,$$

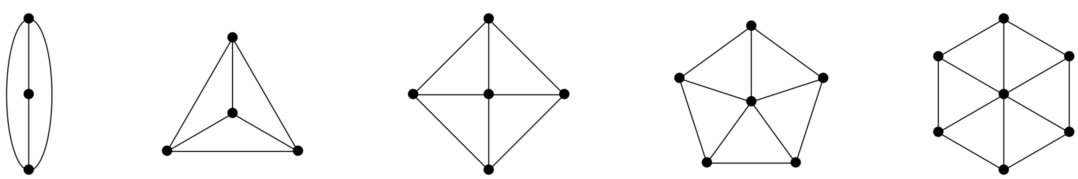
which contradicts the 3-connectedness of M. So $g \in B$. In fact, g is even further from A, because if $g \in \mathrm{cl}(A)$, then $\lambda_M(A \cup \{ e, g \}) = \lambda_{M\backslash e}(A) = 1$, again contradictory unless $|B| = 2$. And if that were the case, $B = \{ g, \text{coloop} \}$, and $M\backslash e$ cannot have any coloops.

So $g \in B - \mathrm{cl}(A)$. By symmetry, $f \in A - \mathrm{cl}(B)$. Since $|M| \geq 8$, up to symmetry we may also assume $|A| \geq 4$.

$$\lambda_{M/f}(B \cup \{e\}) = \lambda_M(B \cup \{e\}) - \sqcap_M(\{f\}, B \cup \{e\}) = \lambda_M(B \cup \{e\}) - 1 \leq \lambda_{M\backslash e}(B) + 1 - 1 = \lambda_{M\backslash e}(B),$$

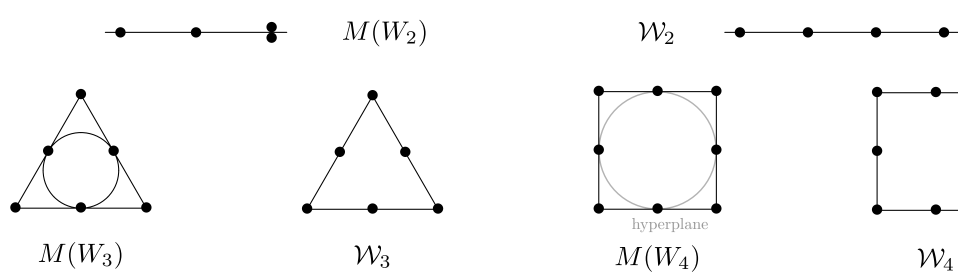
so $(A - \{f\}, B \cup \{e\})$ is a 2-separation in M/f, meaning M/f is not internally 3-connected. By Bixby's Lemma, $M\backslash f$ is internally 3-connected, and since it is not 3-connected by assumption, f is in a triad T^*. T is a circuit and T^* is a cocircuit, so $|T \cap T^*| \neq 1$. But $e \notin T^*$, so $g \in T^*$. By the previous lemma, $M\backslash e$ is 3-connected, a contradiction. □

What we have now found is that, either there is an element $e \in E(M)$ such that one of M/e, $M\backslash e$ is 3-connected, or every element is in both a triangle and a triad—the next course of action is to characterize these matroids.



Let W_n be the **wheel graph** obtained by taking a circuit graph (called the *rim*) and adding a vertex (called the *hub*) adjacent to every vertex (the new edges called *spokes*). The **wheel** of rank n is $M(W_n)$. Observe that the rim is both a circuit and a hyperplane. The **whirl** of rank n, $\mathcal{W}_n$, is the matroid obtained from $M(W_n)$ by **relaxing** the rim—declaring the circuit-hyperplane independent.

Exercise. Let $M = (E, \mathcal{I})$ be a matroid and $C \subseteq E$ a circuit-hyperplane. Show that $(E, \mathcal{I} \cup \{C\})$ is a matroid.



Matroid diagrams of the wheels and whirls of ranks 2, 3, and 4

10 Jul **Lemma**. *Let M be a 3-connected matroid with $|M| \geq 4$. If each element of M is contained in both a triangle and a triad, then M is either a wheel or a whirl.*

Proof (Sketch). Recall that if C is a circuit and C^* is a cocircuit, then $|C \cap C^*| \neq 1$.

Claim 1. $|M| \geq 7$—throw out all the small cases.

Claim 2. M has no $U_{2,4}$-restriction. *Hint*: Look for a 1-separation if it does.

Now, let $B \subseteq E(M)$ be the elements in two triangles of M, and let $B^* \subseteq E(M)$ be the elements in two triads.

Claim 3. For each triangle T, $|T \cap B^*| = 1$, and dually, for each triad T^*, $|T^* \cap B| = 1$.

Claim 4. (B, B^*) is a partition of $E(M)$.

Claim 5. B is a basis.

Claim 6. The fundamental graph $\mathrm{FG}(M, B)$ is a circuit.

Claim 7. M is a wheel or a whirl.

The details are left as an exercise. □

Tutte's Wheels and Whirls Theorem. *Let M be a 3-connected matroid with $|M| \geq 4$. If M is not a wheel or a whirl, there exists an element $e \in E(M)$ such that either M/e or $M\backslash e$ is 3-connected.* □

Corollary. *If M is a 3-connected matroid with $|M| \geq 4$, then there exists a sequence $(N_0, ..., N_k)$ where N_0 is a wheel or a whirl, $N_k = M$, and for each $i \in \{1, ..., k\}$, N_{i-1} is either N_i/e or $N_i\backslash e$ for some $e \in E(M)$.* □

Exercise. Let M be a 3-connected matroid with $|M| \geq 4$. Show there exists $e \in E(M)$ such that the simplification of M/e is 3-connected.

6 Graphic Matroids

13 Jul Recall from §1.3: if G is a planar graph, then $M(G)^* = M(G^*)$. Armed with 3-connectivity, we investigate and exploit this connection further.

Lemma. *If $G = (V, E)$ and $H = (W, E)$ are graphs with the same edge set, $M(G)^* = M(H)$, and $M(G)$ is connected, then there exist planar embeddings of G and H such that $G^* = H$.*

Proof. Since $M(G)$ is connected and $M(G)^* = M(H)$, both $M(G)$ and $M(H)$ are loopless and 2-connected. The result is trivial if $E = \varnothing$, so proceed by induction on $|E|$—let $e \in E$. At least one of $M(G/e)$ and $M(G\backslash e)$ is connected, so by possibly swapping G and H, we may assume $M(G\backslash e)$ and $M(H/e)$ are connected.

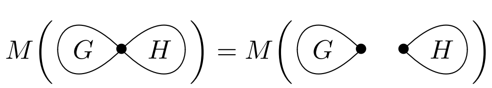
By the inductive hypothesis, there exist planar embeddings of $G\backslash e$ and H/e. Suppose $e = uv$ in H, and that these identify to a vertex w in H/e. Let $C_u = \delta_H(u)$, $C_v = \delta_H(v)$, and $C = \delta_{H/e}(x)$. C_u and C_v are cocircuits of $M(H)$, and C is a cocircuit of $M(H/e)$, because H and H/e are 2-connected graphs.

In the dual embedding of $G\backslash e$, C is a facial circuit, since C is a vertex neighbourhood in H/e. Also, C_u and C_v are circuits of $M(G\backslash e)$ with the property that $C = C_u \cup C_v - \{e\}$. e must be a chord of C in G, and this gives a consistent extension of $G\backslash e$ and H/e to G and H in terms of planar embeddings. □

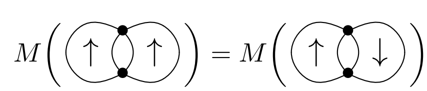
Corollary (Whitney). *A graph G is planar iff $M(G)^*$ is graphic.* □

As a consequence of this, testing whether a matroid is graphic generalizes testing the planarity of a graph.

Does a graphic matroid determine its graph? The answer is no, even if the matroid is connected.



splitting cut-vertices

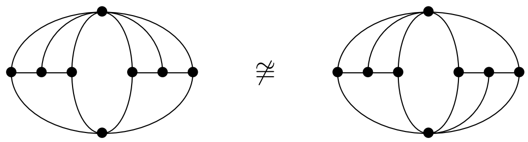


Whitney flips

If the matroid is not 3-connected, the existence of a small separation can generally be exploited to give two different graphs, while preserving all circuits and thus preserving the cycle matroid. If there is a 1-separation, there is a cut vertex, and that vertex may be split.

If there is a 2-separation, then a **Whitney flip** operation can be performed, flipping one side of the separation so as to reverse that portion of each circuit that crosses the separation. As sets, the circuits are still equal, so the cycle matroid does not change, though the cyclic order changes, so the graph may change radically.

Say that a cocircuit C of a matroid M is **separating** if $M\backslash C$ is disconnected.



A Whitney flip relating two nonisomorphic graphs

Proposition. *If $G = (V, E)$ is a 3-connected loopless graph, then a cocircuit $C \subseteq E$ of $M(G)$ is nonseparating iff C is a vertex neighbourhood—$C = \delta_G(v)$ for some $v \in V$.*

Proof. For any $v \in V$, since G is 3-connected, $G - v$ is 2-connected, so $\delta(v)$ is a nonseparating cocircuit. Conversely, if C is a nonseparating cocircuit, $C = \delta(X)$ for some $X \subseteq V$, and then $G\big|X$ and $G\big|(V - X)$ are connected. Since C is nonseparating, one side has no edges, so either X or $V - X$ is a singleton. □

15 Jul **Theorem**. *If G and H are loopless graphs and G is 3-connected, then $M(G) = M(H)$ iff G is equal to H up to vertex relabelling.*[11]

Proof. Since $M(G) = M(H)$, the Proposition implies that for each $v \in V(G)$, there is a $w \in V(H)$ such that $\delta_G(v) = \delta_H(w)$, and vice versa. It can thus be determined that the map sending $v \mapsto w$ is a vertex relabelling. □

This result implies that, in a way, every 3-connected loopless planar graph has a unique planar embedding—vertex neighbourhoods and facial circuits are determined by the cycle matroid and hence by the underlying graph.

6.1 Graphicness Testing

Now, we consider the problem of recognizing whether a matroid is graphic, and then doing so efficiently if it is possible. Knowing that all graphic matroids are binary, and having developed significant tools for binary matroids, we consider first the binary case: *given a binary matroid, is it graphic?*

The following algorithm is an outline of the strategy to determine whether a binary matroid is graphic. We will deconstruct and show each step is feasible, and in fact can be done relatively efficiently.

Algorithm (Is a binary matroid M graphic?).

reduce to the 3-connected case
find sequence $N_0, ..., N_k = M$ of 3-connected minors, such that N_0 is a wheel
inductively try to find G_i so that $M(G_i) = N_i$

Remark. Testing 3-connectedness reduces to computing $\kappa(X, Y)$ for all $X, Y \subseteq E(M)$ with $|X| = |Y| = 2$. While terrible, this is polynomial time, because it can be done with Matroid Intersection.

$$\begin{bmatrix} M_X/\bullet & 0 & 0 \\ \cdots & 1 & \cdots \\ 0 & 0 & M_Y/\bullet \end{bmatrix}$$

To reduce to the 3-connected case, first observe the easy reduction to the connected case: a matroid is graphic iff its components are. Then, consider a 2-separation (X, Y) of a binary $M = M(A)$. Since $\sqcap(X, Y) = 1$, there is a unique nonzero $v \in \mathrm{Col}\big(A\big|X\big) \cap \mathrm{Col}\big(A\big|Y\big)$. Define $M^+ = M([\,A\ v\,])$ with $E(M^+) = E(M) \sqcup \{\bullet\}$. Then let $M_X = M^+\big|(X \cup \{\bullet\})$ and $M_Y = M^+\big|(Y \cup \{\bullet\})$. Up to row operations, M^+ has a representation of the form at left.

Exercise. Show that M has an M_X-minor and an M_Y-minor.

Exercise. Show that M is graphic iff M_X and M_Y are.

Tutte's Wheels and Whirls guarantees the existence of the sequence of successive deletions or contractions ending in a wheel or a whirl—recall that whirls are not graphic, so we require that the smallest minor is a wheel.

Consider the inductive case where M is a 3-connected binary matroid, and $M\backslash e = M(G)$ is 3-connected and graphic for some $e \in E(M)$. The incidence matrix of G is a binary representation of $M\backslash e$, so we may suppose that $M = M([\,G\ t\,])$[12] for some $t \in \mathrm{GF}(2)^{V(G)}$. Let T be the support of t. Every binary extension of a graphic

[11] Vertex relabelling is a stronger condition than isomorphism, because we claim that $M(G)$ and $M(H)$ are *equal* as matroids, not simply isomorphic. Hence: **Corollary**. *If G and H are loopless graphs and one is 3-connected, then $M(G) \cong M(H)$ iff $G \cong H$.*

[12] Due in equal parts to convenience and indolence, identify a graph with its incidence matrix when it is used as a submatrix.

matroid $M(G)$ may be described by the pair (G, T), where G is the graph and $T \subseteq V(G)$—call this pair a **graft**, write $M(G, T)$ for the extended matroid, and call the new element e the **graft element**.

Lemma. *If (G, T) is a graft and $|T|$ is odd, then the graft element is a coloop.*

Proof. Every column of an incidence matrix sums to zero, so if $|T|$ is odd, the column rank of the representation of $M(G, T)$ is increased. From basic linear algebra, column rank equals row rank, so $r(M(G, T)) > r(M(G))$. □

Lemma. *Let (G, T) be a graft such that G is 3-connected and $|T|$ is even. Then $M(G, T)$ is graphic iff $|T| \leq 2$.*

Proof. One direction is clear, so suppose $M(G, T) = M(H)$ is graphic. We may assume $M(G, T)$ is loopless. Then, by the previous Theorem, WLOG $H \backslash e = G$. Then $M(H) = M([\,H\backslash e\ t\,]) = M([\,G\ \chi_T\,])$ so $|T| \leq 2$. □

17 Jul Now, consider the inductive case where M is 3-connected and binary, and $M/e = M(G)$ is 3-connected and graphic for some $e \in E(M)$. This is not covered by the dual of the previous case, because duals of graphic matroids are not necessarily graphic. As before, the representation of M/e will have an extra row added to the incidence matrix of G, but due to the properties of contraction—recall §1.4—there will also be an extra row. Without loss of generality, $M = M\left(\left[\begin{smallmatrix}1 & s^\mathsf{T}\\ 0 & G\end{smallmatrix}\right]\right)$, for some $s \in \mathrm{GF}(2)^{E(G)}$. Let Σ be the support of s.

Every binary coextension of a graphic matroid $M(G)$ may be described by the pair (G, Σ) for $\Sigma \subseteq E(G)$—this is a **signed graph**, with Σ the **signature** or **signing** and its elements the **odd** edges. Write $M(G, \Sigma)$ for the coextended matroid.

Lemma. *Let $\Sigma, \Sigma' \subseteq E(G)$. Then $M(G, \Sigma) = M(G, \Sigma')$ iff $\Sigma \bigtriangleup \Sigma'$ is a cut of G.*

Proof. Two matroids are equal if they have the same rowspace. Observe that $\mathrm{Row}\left(\left[\begin{smallmatrix}1 & s^\mathsf{T}\\ 0 & G\end{smallmatrix}\right]\right) = \mathrm{Row}\left(\left[\begin{smallmatrix}1 & t^\mathsf{T}\\ 0 & G\end{smallmatrix}\right]\right)$ iff $s + t \in \mathrm{Row}(G)$. The support of $\chi_\Sigma + \chi_{\Sigma'}$ is $\Sigma \bigtriangleup \Sigma'$, and $\mathrm{Row}(G)$ is the cocycle space, i.e. the space of cuts. □

Lemma. *Let $e = xy \in E(G)$ and $\Sigma = \delta(x) - \{e\}$. Then $M(G) = M(G/e, \Sigma)$.*

Proof. $G = \begin{array}{c c} & \begin{array}{cccc} e & \Sigma & & \end{array} \\ \begin{array}{c} x \\ y \\ \\ \end{array} & \begin{bmatrix} 1 & \mathbf{1}^\mathsf{T} & 0 & 0 \\ 1 & 0 & \mathbf{1}^\mathsf{T} & 0 \\ 0 & * & * & * \end{bmatrix} \end{array} \sim \begin{array}{c c} & \begin{array}{cccc} e & \Sigma & & \end{array} \\ \begin{array}{c} x \\ x{+}y \\ \\ \end{array} & \begin{bmatrix} 1 & \mathbf{1}^\mathsf{T} & 0 & 0 \\ 0 & \mathbf{1}^\mathsf{T} & \mathbf{1}^\mathsf{T} & 0 \\ 0 & * & * & * \end{bmatrix} \end{array} = (G/e, \Sigma)$, so the matroids are equal. □

A circuit of C of G is **Σ-odd** (**Σ-even**) if $|C \cap \Sigma|$ is odd (even, respectively). Equivalently, say C is an **odd** or **even** circuit of the signed graph (G, Σ). The parity of a circuit is not affected by resigning across a cut. Letting $\bullet$ be the coextension element of $M(G, \Sigma)$ and C a circuit of G, then if C is Σ-even, C is a circuit of $M(G, \Sigma)$, while if C is Σ-odd, $C \cup \{\bullet\}$ is a circuit. Also, say that $v \in V(G)$ is a **blocknode** of (G, Σ) if it is contained in every odd circuit of (G, Σ).

Theorem. *Let (G, Σ) be a signed graph where G is loopless and 3-connected. Then $M(G, \Sigma)$ is graphic iff (G, Σ) has a blocknode.*

Proof. Suppose $M(G, \Sigma) = M(H)$ is graphic. Let v be an end of the coextension element $\bullet$ and let $\Sigma' = \delta(v) - \{\bullet\}$. By the previous lemma, $M(H) = M(H/\bullet, \Sigma')$. Then $M(G) = M(H/\bullet)$, so $G \cong H/\bullet$ since G is loopless and 3-connected. Then $(G, \Sigma) = (H, \Sigma')$, and up to resigning we may assume $\Sigma = \Sigma'$. Hence, (G, Σ) has a blocknode, namely v. Conversely, suppose (G, Σ) has a blocknode v. $(G, \Sigma) - v$ has no odd circuits, so by resigning we may assume each edge in Σ is incident at v. By the previous lemma, $M(G, \Sigma)$ is graphic. □

Remark. The existence of a blocknode can be checked efficiently—for each vertex $v \in V(G)$, determine whether the coextension is a coloop of $M(G, \Sigma) \backslash \delta_G(v)$.

This settles the procedure determining whether a binary matroid is graphic. We now expand our purview to all matroids. Given a matroid, it is generally difficult to determine whether it is binary. However, as it will turn out, determining whether it is graphic is not that much harder than determining if a binary matroid was graphic.

Algorithm (Is a general matroid M graphic?).
find a basis B and the associated fundamental matrix $F = F(M, B)$
determine whether the binary matroid $M([\,I\ F\,])$ is graphic
if $M([\,I\ F\,]) = M(G)$, test whether $M = M(G)$

Recall from §2 that if M is a matroid and B is a basis, then, letting $F = F(M, B)$ be the fundamental matrix, M is binary iff $M = M([I\ F])$. The latter matroid is binary, so we can test whether it is graphic as before. Then if $M([I\ F]) = M(G)$, we may verify whether M is graphic by seeing whether $M = M(G)$. This works because M is necessarily binary if it is graphic.

Hence, we reduce the problem to testing whether $M = M(G)$, given a matroid M and a graph G. With the following result, we obtain that in polynomial time.

Theorem (Seymour). *Given a matroid M and a connected graph G, $M = M(G)$ iff $r(M) = |V(G)| - 1$ and for each $v \in V(G)$ and $e \in \delta(v)$, there is a cocircuit C of M with $e \in C \subseteq \delta(v)$. Proof. Exercise.* □

6.2 Tutte's Excluded Minors

20 Jul Using the tools developed above, we may determine the excluded minors for the class of graphic matroids, thereby proving a result of Tutte advertised earlier. $U_{2,4}$ is an excluded minor for both the classes of binary and graphic matroids, so it suffices to classify the collection of *binary* excluded minors for the graphic matroids—call this $\mathcal{E}$.

We will proceed by looking at the behaviour of $M \in \mathcal{E}$ under the graphicness testing algorithm developed above. M must be 3-connected, because if it is not, then one of its 3-connected blocks also fails to be graphic. M cannot be a wheel, since wheels are graphic, and it cannot be a whirl, since whirls are not binary. Thus, there exists some $e \in E(M)$ such that one of $M \backslash e$ or M/e is 3-connected and graphic, and we can describe M by either a graft or a signed graph.

Recall a graft is a pair (G, T) with G a graph and $T \subseteq V(G)$. If $|T|$ is odd, the graft element is a coloop, so $M(G, T)$ is graphic. If $|T|$ is even, then $M(G, T)$ is graphic iff $|T| \leq 2$. Like graphs and matroids, grafts also have a sensible notion of minor which we can consider—we will observe the behaviour of the extended matroid under minors to make sense of it.

If $e = uv \in E(G)$ then $M(G, T)\backslash e = M(G\backslash e, T)$ clearly. Suppose also that $M(G, T)/e = M(G/e, T')$ for some T'. Let $u \oplus v$ be the vertex obtained by contracting uv in G. If $u, v \notin T$ then it stands to reason that $u \oplus v \notin T'$—in fact, $T' - \{u \oplus v\} = T - \{u, v\}$, so only u and v affect whether $u \oplus v \in T'$. If, say, $u \in T$ but $v \notin T$, then $u \oplus v \in T'$. On the other hand, if both $u, v \in T$, then $u \oplus v \notin T$.

Define $(G, T)\backslash e := (G\backslash e, T)$ and $(G, T)/e := (G/e, T')$, for $T' \subseteq V(G/e)$ such that $T' - \{u \oplus v\} = T - \{u, v\}$ and $|T| \equiv |T'| \bmod 2$. It can be verified this interacts with matroid extension in the way described above.

Now suppose that $M \in \mathcal{E}$ has an $e \in E(M)$ such that $M\backslash e$ is 3-connected. Then $M = M(G, T)$ for some simple 3-connected graph G and $T \subseteq V(G)$ such that $|T|$ is even and at least 4.

Claim (a). *If $f = uv \in E(G)$ such that G/f is 3-connected, then $|T| = 4$ and $u, v \in T$.*

Proof. $M(G, T)/f = M(G/f, T')$ for $|T'| \equiv |T| \equiv 0 \bmod 2$ and hence $|T'| \leq 2$. Since $T' - \{u \oplus v\} = T - \{u, v\}$, it must be the case that $|T| = 4$ and $u, v \in T$. □

Since $|M(G)| \geq 4$, we may use an exercise from §5.2, so there exists an $f \in E(G)$ such that the simplification of $M(G)/f$ is 3-connected. Then G/f is a 3-connected graph, so we deduce that $|T| = 4$ by Claim (a).

22 Jul **Claim** (b). *If $f = uv \in E(G)$ and $u \notin T$, then $v \in T$, $\deg(v) = 3$, and $N(v) - \{u\} \subseteq T$.*

Proof ($*$). G/f cannot be 3-connected, by Claim (a), so $M(G)/f$ is not internally 3-connected. By Bixby's Lemma, $M(G)\backslash f$ must be internally 3-connected. Deletion in a 3-connected graph can introduce at most two series pairs, so there are at most two vertices of degree 2, and these vertices are a subset of $\{u, v\}$. If w is such a vertex, then consider $f' = ww' \in E(G) - \{f\}$—the graft minor $(G', T') = (G, T)\backslash f/f'$ has $|T'| = |T| = 4$ unless both $w, w' \in T$. This is undesirable, since $M\backslash f = M(G, T)\backslash f$ is graphic. So unless $\deg_{G\backslash f}(v) = 2$, $v \in T$, and $N_{G\backslash f}(v) \subseteq T$, we can obtain a non-graphic minor of $M\backslash f$. □

If we now suppose $|V(G)| \geq 6$, there exist distinct $u, v \notin T$. They cannot be adjacent, for fear of contradicting Claim (b). But then $N(u), N(v) \subseteq T$ and $N(u) \cap N(v) = \varnothing$, so $4 = |T| = |N(u)| + |N(v)| \geq 3 + 3 = 6$, again a contradiction. So $|V(G)| \leq 5$.

There are no 3-connected graphs on three vertices or fewer. The only 3-connected graph on four vertices is K_4, and then $T = V(K_4)$ by necessity. Then

$$M = M(K_4, V(K_4)) = M\left(\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}\right) = \mathrm{PG}(2,2) = F_7,$$

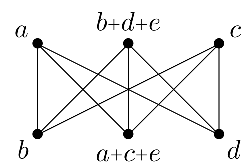
since the last row of $[\,G\ \chi_T\,]$ is the sum of the other three.

Else, $|V(G)| = 5$. Since there are an odd number of vertices, there is a vertex u of even degree, and since G is 3-connected, $\deg(u) = 4$. By Claim (b), $u \notin T$, so then $\deg_{G-u}(v) = 2$ for all $v \in T$, and since G is simple, $G - u$ is the circuit on 4 vertices. Hence $G = W_4$, and $T = V(W_4) - \{u\}$ is every vertex except the hub. Then,

$$M = M\left(\begin{bmatrix} 1 & & & & 1 & 0 & 0 & 1 & 1 \\ & 1 & & & 1 & 1 & 0 & 0 & 1 \\ & & 1 & & 0 & 1 & 1 & 0 & 1 \\ & & & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right) = M\left(\begin{bmatrix} 1 & & & & & 1 & 1 & 0 & 0 \\ & 1 & & & & 0 & 1 & 1 & 0 \\ & & 1 & & & 0 & 0 & 1 & 1 \\ & & & 1 & & 1 & 0 & 0 & 1 \\ & & & & 1 & 1 & 1 & 1 & 1 \end{bmatrix}\right)^*,$$

which is a representation of $K_{3,3}$, so $M = M(K_{3,3})^*$. This exhausts the cases.

$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{array}{c} a \\ b \\ c \\ d \\ a{+}c{+}e \\ b{+}d{+}e \end{array}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$$



Theorem. *If $M \in \mathcal{E}$ has $e \in E(M)$ such that $M\backslash e$ is 3-connected, then M is either F_7 or $M(K_{3,3})^*$.* $\square$

Recall a signed graph is a pair (G, Σ) with G a graph and $\Sigma \subseteq E(G)$ a collection of edges declared odd, which can be freely resigned by taking a symmetric difference with any cut. $M(G, \Sigma)$ is graphic iff (G, Σ) has a blocknode, a vertex contained in each Σ-odd circuit of the signed graph. We now develop minors of signed graphs.

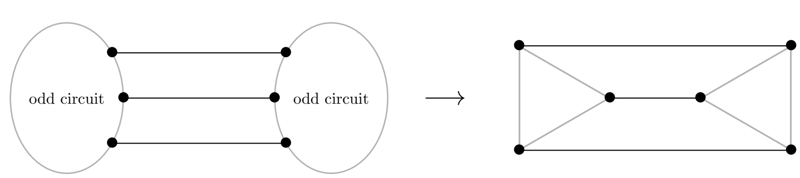
Let $e = uv \in E(G)$. Easily, $M(G,\Sigma)\backslash e = M(G\backslash e, \Sigma - \{e\})$. Also, if $e \notin \Sigma$, then $M(G,\Sigma)/e = M(G/e, \Sigma)$. If $e \in \Sigma$ and $u \neq v$, we may simply resign Σ across either of $\delta(u)$ or $\delta(v)$ and then contract as before, so $M(G,\Sigma)/e = M(G, \Sigma \mathbin{\triangle} \delta(u))/e = M(G/e, \Sigma \mathbin{\triangle} \delta(u))$. If $u = v$, then e is parallel to the coextension and to avoid figuring out coherent behaviour, we will simply disallow loop contraction.

Given the above, declare that a signed graph (G', Σ') is a minor of (G, Σ) if there exist disjoint $C, D \subseteq E(G)$ and an $X \subseteq V(G)$ such that $G' = G\backslash D/C$, $\Sigma' = \Sigma \mathbin{\triangle} \delta(X)$, and $C \cap \Sigma' = \varnothing$.

24 Jul **Lemma**. *If G is a simple 3-connected graph and the signed graph (G, Σ) has two disjoint odd circuits, then $M(G, \Sigma)$ has a F_7^*–minor or $M(K_5)^*$–minor.*

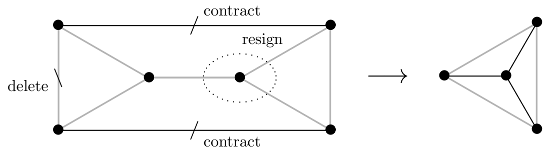
Proof. By Menger's Theorem, there exist three disjoint paths between the odd circuits, so by contracting the circuits down to odd triangles, the paths down to single edges, and deleting everything else, (G, Σ) has a minor $(\overline{C}_6, \Sigma')$ on the triangular prism graph $\overline{C}_6$. Without loss of generality, the two triangles can be made all odd, since the circuits were odd. After that, the three middle edges are a cut, so they can be resigned to have either one or three edges odd.

$(G, \Sigma) =$



$= (\overline{C}_6, \Sigma')$

If one is odd, then by contracting the remaining two and deleting an edge from the resulting parallel pair, $(\overline{C}_6, \Sigma')$ has the minor $(K_4, E(K_4))$, and since vertex neighbourhoods are cuts, $M(K_4, E(K_4)) = M(K_4, E(K_3)) = F_7^*$.



$$\longrightarrow \quad M\left(\begin{bmatrix} 1 & & & 0 & 0 & 1 & 1 \\ & 1 & & 0 & 1 & 0 & 1 \\ & & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}\right) \quad = \quad F_7^*$$

If all three are odd, then $M(\overline{C}_6, \Sigma') = M(K_5)^*$. (*Exercise.*) Hence $M(G, \Sigma)$ has F_7^* or $M(K_5)^*$ as a minor. □

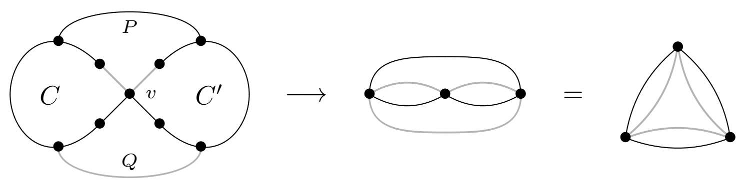
Clearly, a signed graph cannot at once have disjoint odd circuits and a blocknode. It would be ideal if these conditions were exhaustive as well as mutually exclusive, but as we will see, this is not the case. Let (G, Σ) be a signed graph such that G is simple and 3-connected, having no blocknode and no two circuits disjoint.

Claim (a). *Let (H, Σ') be a subgraph of (G, Σ) and $v \in V(H)$ such that (H, Σ') has an odd circuit and Σ' can be resigned to a subset of $\delta_H(v)$. Then either* (1) *there is an odd circuit C of G such that $|V(C) \cap V(H)| = 1$ and $v \notin C$, or* (2) *there is an* ***odd ear****—an odd path P of (G, Σ) with ends $\{u, w\} = V(P) \cap V(H) \subseteq V(H) - \{v\}$.*

Proof. Since (G, Σ) has no blocknode, there exists an odd circuit C avoiding v, and since (G, Σ) has no disjoint odd circuits, C must intersect H. If $|V(C) \cap V(H)| = 1$, we are done, so suppose it is at least 2. Up to resigning, $(H, \Sigma') - v$ has no odd edges, so one of the ears produced by C must be odd. □

Let C and C' be odd circuits of (G, Σ). They cannot be disjoint, so suppose $V(C) \cap V(C') = \{v\}$. Let $(H, \Sigma') = C \cup C' \cup P$, where P is a shortest $(V(C) - v, V(C') - v)$-path in $(G, \Sigma) - v$. By resigning, $\Sigma' \subseteq \delta(v)$. Then apply Claim (a). If we obtain outcome (1), then the resultant circuit cannot be attached to v, so it is disjoint from at least one of C or C' and there is a contradiction.

So suppose we obtain (2)—an odd ear Q of (H, Σ'). If it is not the case that one end of Q is in $V(C) - v$ and the other is in $V(C') - v$, then Q will be part of an odd circuit disjoint from whichever of C or C' it does not intersect. On the other hand, if it is the case, then we may contract $C - v$ and $C' - v$ down to vertices, P to an even edge, and Q to an odd edge, to obtain a minor in the form of a "doubled-triangle"—a triangle made of parallel pairs, where exactly one member of each pair is odd. We call this signed graph $(KK_3, E(K_3))$.



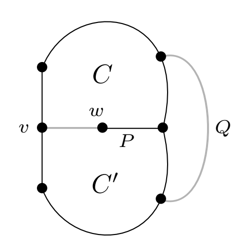
Claim (b). *If there exist odd circuits C and C' with $|V(C) \cap V(C')| = 1$, (G, Σ) has a $(KK_3, E(K_3))$-minor.* □

27 Jul Now we may suppose that for all pairs of odd circuits C and C', $|V(C) \cap V(C')| \geq 2$. Observe that this precludes outcome (1) of Claim (a).

Claim (c). *There exist odd circuits C and C' such that $C \cap C'$ is a path.*

Proof. Take any odd circuit C and vertex $v \in V(C)$, and apply Claim (a) to $(C, \Sigma \cap E(C))$ and v to obtain an odd ear P avoiding v. Let C' be the unique circuit in $(C - v) \cup P$. C' is odd and $C \cap C'$ is a path. □

Now consider a pair of circuits C and C' that have $C \cap C' =: P$ a path and minimize $|V(P)|$. Let v be an end of P, and resign (G, Σ) so that the only odd edge of $C \cup C'$ is the edge of P incident at v. By Claim (a) we obtain an odd ear Q avoiding v. Q cannot have any ends in P—as it would contradict the minimality of P—and hence, for fear of introducing disjoint circuits, must have one end in $V(C) - V(P)$ and the other in $V(C') - P$.



Then $C \cup C' \cup Q$ can be contracted down to K_4, on the vertex set $\{v, P - v, C \cap Q, C' \cap Q\}$. The odd edges comprise the matching $\{P, Q\}$, so the graph can be resigned into $(K_4, E(K_4))$.

Theorem. *Let G be a simple, 3-connected graph and let (G, Σ) be a signed graph having no blocknode and no two circuits disjoint. Then (G, Σ) has $(K_4, E(K_4))$ or $(KK_3, E(K_3))$ as a minor.* □

Corollary. *If $M \in \mathcal{E}$ has $e \in E(M)$ such that M/e is 3-connected, then M is either $M(K_5)^*$, F_7^*, or F_7.*

Proof. It remains to observe $M(KK_3, E(K_3)) = F_7 = M(K_4, E(K_4))^*$ and $M(\overline{C}_6, E(\overline{C}_6)) = M(K_5)^*$. □

This concludes Tutte's excluded minor characterization of the graphic matroids.

Theorem (Tutte). $\mathcal{E} = \{ F_7, F_7^*, M(K_5)^*, M(K_{3,3})^* \}$. □

-1 The End

The remainder of these notes will cover bonus content—extra material, alternative proofs, interesting applications, what can come next; anything I find too good to leave without mention—as my time and sanity permits. Under constant construction, *caveat lector*, and so on. Latest update: 2 June 2016.

-1.1 Regular Matroids

Recall from §1.4 that a matroid is **regular** if it is representable over every field. This class of matroids sits somewhere between the binary and graphic matroids. We present without proof some important results regarding regular matroids, beginning with results presented earlier.

Theorem (Tutte). *For a matroid M, the following are equivalent:*

(1) *M is regular.*

(2) *M is representable over* GF(2) *and* GF(3)*.*

(3) *M is representable over* GF(2) *and some field of characteristic $\neq 2$.*

(4) *M is representable over $\mathbb{R}$ by a totally unimodular matrix.*

Notably new in the above theorem is that there is nothing special about GF(3)-representability once the matroid is already binary. Indeed, every regular matroid has a unique representation in every field, up to row operations and reordering columns.[13]

The regular matroids are closed under minors. The excluded minors (due to Tutte) are $U_{2,4}$, F_7, and F_7^*.

A matroid M is a **k-sum**[14] $M_1 \oplus_k M_2$ of matroids M_1 and M_2 if there exists an **exact k-separation** of M—that is, a k-separation (E_1, E_2) with $\lambda(E_1) = k-1$—and M can be constructed by combining M_1 and M_2 together across that separation. We will only need to understand k-sums for $k \leq 3$, so this general definition will be left intentionally vague.

If M_1 and M_2 are matroids, then their **1-sum** $M_1 \oplus_1 M_2$ is simply their direct sum $M \oplus N$. If A and B are matrices over the same field, then $M(A) \oplus M(B) = M([\begin{smallmatrix} A & \\ & B \end{smallmatrix}])$.

Note that the direct sum need not preserve representability in general. F_7 is binary but is an excluded minor of the regular matroids, so by the equivalence theorem given above, it must be non-representable over every other field. Relaxing a circuit-hyperplane (§5.2) in F_7 gives F_7^-, the **non-Fano matroid**, and it is only representable in fields of characteristic not equal to 2 (*Exercise.*) so $F_7 \oplus F_7^-$ is not representable.

If M_1 and M_2 are matroids, with $e_1 \in E(M_1)$ and $e_2 \in E(M_2)$, then let M be the matroid given by freely extending e into $\mathrm{cl}(\{e_1, e_2\})$. Then $M/e\backslash\{e_1, e_2\}$ is a **2-sum** of M_1 and M_2. For regular matroids,

$$M\left(\begin{bmatrix} A & \\ a & 1 \end{bmatrix}\right) \oplus_2 M\left(\begin{bmatrix} 1 & b \\ & B \end{bmatrix}\right) = M\left(\begin{bmatrix} A & \\ a & b \\ & B \end{bmatrix}\right).$$

If M_1 and M_2 are matroids, with F_1 a line of M_1 and F_2 a line of M_2 such that $F_1 \cong F_2$, their **3-sum** is constructed by identifying these lines and then deleting them from the matroid. For regular matroids,

[13] Was someone recorded to have proven this? Is it 'folklore'? I don't know. It's not that difficult to show.

[14] A k-sum is not unique given its 'summands', so an instance of $X \oplus_k Y = Z$ is an abuse of notation—its accurate interpretation is that Z has a *k-sum decomposition* into X and Y.

$$M\left(\begin{bmatrix} A & \\ A' & \begin{smallmatrix}1&0&1\\0&1&1\end{smallmatrix}\end{bmatrix}\right) \oplus_3 M\left(\begin{bmatrix} \begin{smallmatrix}1&0&1\\0&1&1\end{smallmatrix} & B' \\ & B\end{bmatrix}\right) = M\left(\begin{bmatrix} A & \\ A' & B' \\ & B\end{bmatrix}\right).$$

The regular matroids are closed under k-sums for $k \leq 3$, as the examples above imply.

Seymour's Decomposition Theorem. *Every regular matroid can be constructed by k-sums, $k \leq 3$, from graphic matroids, cographic matroids, and the matroid R_{10}.*

$$R_{10} := M\left(\begin{bmatrix} 1 & & & & & -1 & 1 & 0 & 0 & 1 \\ & 1 & & & & 1 & -1 & 1 & 0 & 0 \\ & & 1 & & & 0 & 1 & -1 & 1 & 0 \\ & & & 1 & & 0 & 0 & 1 & -1 & 1 \\ & & & & 1 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}\right) = M(K_{3,3}, \text{perfect matching}).$$

Note that R_{10} is self-dual, and neither graphic nor cographic.

-1.2 Matroid Resources

A short list of books and other resources on matroids or of interest to those interested in matroids.

The Matroid Union (`http://matroidunion.org/`) is a semi-active math blog for the matroid community. It contains great expository posts by world-class matroid theorists on various facets of matroids and their study.

James Oxley's *Matroid Theory* is the recommended text for the course, and for a good reason: it is a good, comprehensive resource running the gamut of matroid theory.

Klaus Truemper freely distributes electronic copies of his book *Matroid Decomposition*, a 400-page tome on various decompositions of graphs and matroids. (`https://www.utdallas.edu/~klaus/Mbook/`)

Greedoids by Korte, Lovász and Schrader deals with more general set-systems than matroids, to investigate their power in relation to matroids.

There is a book called *Coxeter Matroids* by Borovik, Gelfand, and White, dealing with some sort of crazy geometric generalization of matroids. (This is more a reminder for me than for you.)

Two books in Rota's *Encyclopedia of Mathematics and its Applications* series, namely *Theory of Matroids* and *Combinatorial Geometries* (both edited chiefly by Neil White) have proven invaluable. Worth mentioning is an appendix of the former, by Thomas Brylawski: it gives a meticulous (though amazingly, by no means complete—and this is a compliment to matroid theory!) account of the many **cryptomorphisms**[15] of matroids.

-1.3 Bose–Burton Revisited

Recall the Bose–Burton Theorem (§3.1), characterizing binary rank-r $\mathrm{PG}(t-1,2)$-free matroids: they may have size at most $\mathsf{ex}(t,r) = (1-2^{1-t})2^r$, and at equality must be isomorphic to the Bose–Burton geometry $\mathrm{BB}(r-1,2,t-1) = \mathrm{PG}(r-1,2)\backslash F$ for some rank-$(r-t+1)$ flat F.

The proof given in these notes is fully-digested for the matroid novice, and the extremal case is left as an exercise, but there is a neat and tidy way to prove it for those who have wrapped their heads around contraction. The argument, presented as part of a *Matroid Union* post by Jim Geelen (`http://matroidunion.org/?p=578`) is reproduced below.

Proof of Bose–Burton. Proceed by induction on t. Suppose X is a set of points of $\mathrm{PG}(r-1,2)$ such that $\mathrm{PG}(r-1,2)\backslash X$ is $\mathrm{PG}(t-1,2)$-free. Contract a point $p \notin X$ and simplify to obtain $\mathrm{PG}(r-2,2)$; let X' be the image of X in this geometry. Then $\mathrm{PG}(r-2,2)\backslash X'$ is $\mathrm{PG}(t-2,2)$-free, so X must have at least

[15] Axiomatizations or characterizations that are equivalent, but not obviously so. Matroid theorists love this word, because it sounds so cool. Other cryptomorphisms can be found in the definitions of the determinant and semidefinite matrices.

$$|X| \geq |X'| \geq 2^{(r-1)-(t-1)+1} - 1 = 2^{r-t+1} - 1$$

elements. Furthermore, if X is a flat, then $\mathrm{PG}(r-1,2)\backslash X \cong \mathrm{BB}(r-1,2,t-1)$; otherwise, we may choose p from $\mathrm{cl}(X) - X$ so that $|X| > |X'|$ and equality cannot hold. □

It is worthwhile mentioning the nonbinary case, as Bose and Burton prove their theorem for all finite fields. The interpretation of 'cocycle' and 'critical number' in general $\mathrm{GF}(q)$-matroids is more involved, but it turns out the Bose–Burton geometry $\mathrm{BB}(r-1,q,c)$ is defined in the same way—by removing a flat of rank $r-c$—and the analogous result holds. The proof, however, has not met the same fate, requiring some flat-counting and intense reasoning. See the first half of their 1966 paper, *A Characterization of Flat Spaces in a Finite Geometry and the Uniqueness of the Hamming and the MacDonald Codes.*[16]

-1.4 Partial Fields

It was stated in §(-1).1 that if a matroid is representable over GF(2) and any field of characteristic other than 2, it is regular. So if a matroid is GF(2)-representable, it falls into one of two classes: binary or regular. Is the situation similar for ternary matroids? Can we go even further?

It has been expounded in many places the motivation and evidence resulting from the above line of inquiry—most exceptionally by Semple and Whittle in their seminal paper, *Partial fields and matroid representation*—but the gist is that the theory of matroid representation is richest and most general when replacing fields by a class of field-like objects called **partial fields**, which are distinguished by having a partial addition.

Let R be a commutative ring[17] and let G be a (multiplicative) subgroup of the units of R such that $-1 \in G$. Then the partial field $\mathbb{P} = (R, G)$ is $G \cup \{0\}$ with multiplication and partial addition inherited from R.

The notion of a partial addition is somewhat tricky, but it can be handled. We inherit commutativity easily from the ring. Then, declare a **preassociation** of a multiset S of elements in a partial field $\mathbb{P}$ to be any expression summing together the elements of S such that every *proper* subexpression is defined, and say an **association** is a preassociation with the final sum defined. We "define" the partial associative law as follows: if S is a multiset and there exists a multiset Z of the form $\{ z_1, -z_1, ..., z_n, -z_n \}$ such that there exists an association of $S \cup Z$ with the sum s, then any preassociation of $S \cup Z'$ for Z' of the same form as Z is an association with the sum s.[18]

At this point the industrious reader may exercise their commutative algebra and conclude that many basic results about rings or fields are true of partial fields as well, so long as care is taken to make sure sums are defined at every stage. There is a notion of homomorphism just as for rings, with the additional stipulation that if a sum exists in the domain, the sum of the images must *exist* in the codomain and equal the image of the sum. Homomorphism is especially useful because every field $\mathbb{F}$ is a partial field, via $(\mathbb{F}, \mathbb{F}^*)$.

The naïve notion of partial linear combination appears initially to work, allowing addition of $\mathbb{P}$-vectors only when the sums in each coordinate are defined, but trouble arises when trying to decide the independence of linear combinations that cannot be summed. Instead, recall the sum-over-permutations definition of the determinant:

$$\det A = \sum_\sigma \mathrm{sign}(\sigma) \prod_i A_{i\,\sigma_i}, \quad \text{over all permutations } \sigma \text{ and indices } i.$$

Say that a matrix $A \in \mathbb{P}^{r \times E}$ is a **$\mathbb{P}$-matrix** if every square submatrix has a defined determinant. Declare a subset $\{ i_1, ..., i_n \}$ of columns from a matrix independent if some $n \times n$ submatrix of $[\, v_{i_1} \ \cdots \ v_{i_n} \,]$ has nonzero determinant. This agrees on fields, so we obtain a column matroid for matrices over partial fields. It is not hard to see that the class of $\mathbb{P}$-representable matroids is closed under duality, minors, direct sums, and 2-sums (§(-1).1).

This is a lot of definition to swallow at once, but we may reap rewards from this new theory of representability almost immediately. Consider the partial field $\mathbb{U} = (\mathbb{Z}, \{\pm 1\})$. Since we require all subdeterminants to exist in $\mathbb{U}$, $\mathbb{U}$-matrices are exactly the totally unimodular matrices (§1.4), and hence it can be said that the class of $\mathbb{U}$-representable matroids is exactly the regular matroids.

[16] The authors both have the initials R. C. B., a coincidence which I found amusing.

[17] It is a not-altogether-complicated exercise in algebra to show that requiring R to be a field produces an equivalent definition.

[18] The purpose of Z and Z' is to stamp out a naughty counterexample—it does not change the spirit of the law, only the letter.

A trivial homomorphism is when the entire domain is in the kernel. An important and very relevant result about partial fields is that, if there exists a non-trivial homomorphism from one partial field to another, then any matroid representable in the first is representable in the second. For instance, $\mathbb{U}$ has a non-trivial homomorphism into every partial field, which explains why total-unimodularity implies representability in every field.

To properly prove or even motivate the following examples would be a monumental task—Actaeon would do well not to say anything, lest he make this forest his home—so instead some important classes of partial-representable matroids and their fields will be listed below with no further investigation.

class	partial field	field-representability that suffices
regular matroids	$\mathbb{U} = (\mathbb{Z}, \{\pm 1\})$	$\mathbb{F}_2$ and any char. $\neq 2$ field
dyadic matroids	$(\mathbb{Q}, \{\pm 2^n \mid n \in \mathbb{Z}\})$	$\mathbb{F}_3$ and any of $\mathbb{F}_5$, $\mathbb{Q}$, $\mathbb{R}$
$\sqrt[6]{1}$-matroids	$(\mathbb{C}, \{a \in \mathbb{C} \mid a^6 = 1\})$	$\mathbb{F}_3$ and $\mathbb{F}_4$
near-regular matroids	$(\mathbb{Q}(\alpha), \{\pm\alpha^m(1-\alpha)^n \mid m, n \in \mathbb{Z}\})$	$\mathbb{F}_3$ and (($\mathbb{F}_4$ and $\mathbb{F}_5$) or $\mathbb{F}_8$)
golden-ratio matroids	as above, but for $\alpha^2 - \alpha - 1 = 0$	$\mathbb{F}_4$ and $\mathbb{F}_5$

Notice for instance that the near-regular matroids fall into the intersection of dyadic matroids, $\sqrt[6]{1}$-matroids, and golden-ratio matroids. The landscape of partial field representation is odd and twisted, but it is also rich, and as might be implied by the following theorem, quite powerful.

Theorem (Vertigan). *Let $\mathcal{F}$ be a collection of fields. Then the class of matroids representable in* (1) *every field of $\mathcal{F}$* (2) *at least one field of $\mathcal{F}$ is the class of matroids representable in some partial field.*

I acknowledge as additional resources a multipart *Matroid Union* post (`http://matroidunion.org/?p=238` and `http://matroidunion.org/?p=1430`) by Stefan van Zwam and slides from a talk of Whittle's given at a 2013 conference (`http://www.maths.qmul.ac.uk/~camconf/talks/whittle.pdf`). The result of Vertigan's is probably unpublished, but appears to follow from the work of Pendavingh and van Zwam, in their paper *Lifts of matroid representations over partial fields* (§5).

-1.5 Tutte Homotopy Theorem?

References: Tutte, *A Homotopy Theorem for Matroids, I* and *II*. White, *Combinatorial Geometries* §3.2.

Note to self: Tutte uses a different definition of 'flat'—Tutte flats are complements of coflats. Use White.

-1.6 Infinite Matroids?

Note to self: Read `http://matroidunion.org/?p=1145` but follow the sources through and bash out the things.

-1.7 Minor-Closed Classes?

Coming soon: my adventures with Dowling geometries and how they relate to the structure of minor-closed classes of matroids.