CO444

# Defining Cayley Graphs

5 Jan 2014

Take a group G. The Cayley graph $X(G,C)$ is defined as follows, for some subset $C \subseteq G$:

- the vertex set of X is G
- $x, y \in G$ are adjacent if $yx^{-1} \in C$

To avoid loops, we assume $1 \notin C$. For the graph to be undirected, $C = C^{-1}$.

For example, $C_5 = X(\mathbb{Z}_5, \{1,4\})$ and $C_n = X(\mathbb{Z}_n, \{1,-1\})$. Cayley graphs on $\mathbb{Z}_n$ are called circulants.

Cayley graphs on $\mathbb{Z}_2^d$ are called cubelike graphs: $X(\mathbb{Z}_2^d, C)$ where $C \subseteq \mathbb{Z}_2^d \setminus 0$. For instance, the 3-cube $Q_3 = X(\mathbb{Z}_2^3, \{e_1, e_2, e_3\})$, and generally the d-cube is $X(\mathbb{Z}_2^d, \{e_i \mid 1 \le i \le d\})$.

Claim. $X(\mathbb{Z}_2^d, C)$ is connected iff C is a spanning subset of $\mathbb{Z}_2^d$.

As another example, consider $G = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle$ and $C = \{a, b\}$. (Figure 1) It is connected, as is $X(\langle C \rangle, C)$.

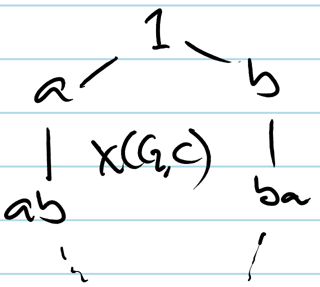

Fig 1

## Automorphisms of Cayley Graphs

If X is a graph, $Aut(X) \le Sym(V(X))$. Subgroups of $Sym(\cdot)$ are called permutation groups. If $X = X(G,C)$ Cayley, we define permutations of G as follows. Given $g \in C$, $\rho_g$ is the permutation of G so that $x \longmapsto xg^{-1}$. $\rho_{gh} = \rho_g \circ \rho_h$ so $g \mapsto \rho_g$ is a homomorphism from G to Sym(G) — a representation.

Also, $g \in G$, define $\lambda_g : G \to G$ by $x \longmapsto gx$. $g \mapsto \lambda_g$ is also a permutation representation, and $\lambda_g$ commutes with $\rho_h$ for all $g, h \in G$.

Claim. $\rho(G) \le Aut(X(G,C))$ Pf. Exer.

If $x, y \in G$, $\rho_{y^{-1}x}$ maps x to y. Hence G acts transitively on itself?

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7 Jan 2014

## Cayley Graph

a group $G$ and a connection set $C \leq G \setminus 1$

$a \sim b$ in the Cayley graph iff $ba^{-1} \in C$

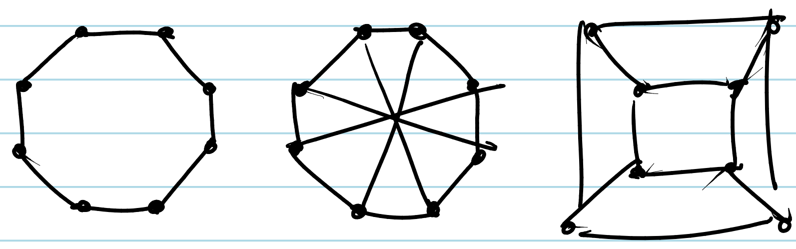
must be vertex transitive (stronger than regular?)

graphs on $n$ vertices that are v.t. have complements (deg $n-k-1$?)

disconnected v.t. graphs have isomorphic v.t. components

nothing surprising for numbers of vertices of 1 to 7
↳ all end up being circulants

when $n=8$:



not a circulant, but cubelike so still Cayley

Exer. Why are ↑ and ↑ not isomorphic?

$Q_3$ would be a circulant if something in $Aut(Q_3)$ had ord $=8$.

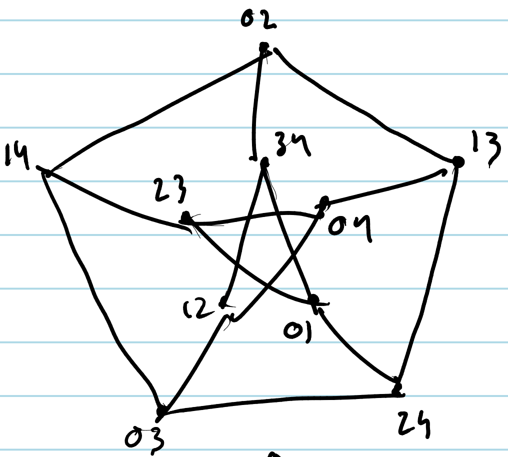
when $n=9$: Caylays for $\mathbb{Z}_3^2$ or $\mathbb{Z}_9$ (valenay 2,4)

all are probably circulants?

when $n=10$: valencies 2,3,4

Claim. Petersen is not a Cayley graph

$Sym(5) \leq Aut(P)$ kinda sorta



Petersen Graph $P$

Groups of order 10: $C_{10}$ cyclic and $D_{10}$ dihedral

$C_{10}$ is abelian so any cubic Cayley on it will have girth ~~at most~~ 4.
→ $P$ not Cayley on $C_{10}$

$D_{2n}$ is the group of symmetries of an $n$-gon

$D_{2n} = \langle r, s \mid r^2 = s^n = rsrs = 1 \rangle = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle$

If conn.set. $C = \{r, s, s^{-1}\}$, you get a 4-cycle, so not Cayley with $(D_{10}, C)$

Try conn.set. $C = \{a, b, c\}$ where $a^2 = b^2 = c^2 = 1$. (Doesn't work.)

Now, $P$ is v.t. but not Cayley.

generalized Johnson graph $J(v,k,i)$

$k$-subsets of $\{1, \ldots, v\}$, $\alpha \sim \beta$ if $|\alpha \cap \beta| = k-i$

if $i = k-1$, plain Johnson graph; $P = J(5,2,0)$

gen. Johnsons are not usually Cayleys

## Perm Grps

Suppose $G$ is a permutation group on $\Omega$.

Define relation $\approx$ on $\Omega$ where $i \approx j$ if $j = i^g$ for some $g \in G$.

$\approx$ is an equiv rel (Exer.) and the eq. classes in $\Omega/\approx$ are the orbits of $G$ on $\Omega$.

The orbits partition $\Omega$ and $G$ is transitive if $|\Omega/\approx| = 1$.

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# Orbit Stabilizer Theorem

Suppose $G$ acts transitively on $\Omega$, i.e. there is always $g \in G$ relating two elements of $\Omega$. Assume $1 \in \Omega$ and define

$$G_{1 \to i} = \{ g \in G \mid 1g = i \} \text{ for } i \in \Omega.$$

These sets partition $G$. $G_1 = G_{1 \to 1} \leq G$ and is called the stabilizer of 1 in $G$. $G_{1 \to i}$ is a coset of $G_1$, of the same size. If $h \in G$ so that $1h = i$ and $g \in G_1$, then $gh \in G_{1 \to i}$ and there exists a map $g \mapsto gh$.

**Theorem.** Orbit Stabilizer Thm. $|G| = |G_1||1G|$ ← orbit of 1 under G

We use $|G:H|$ to denote index i.e. $|G|/|H|$ (not exactly...)

**Remark.** If $G$ acts on $\Omega$, then we have actions also on

- $\Omega \times \Omega$ and $\Omega^k$,
- subsets,
- $k$-subsets,

in the obvious way. If $G$ acts on a graph $X$, it acts also on

- edges
- arcs (ord. pair of adj. vxs)
- s-arcs

Graphs can be edge-transitive but not arc-trans, but are then bipartite (Thm?)

An s-arc is an $(s+1)$-tuple of vertices such that consecutive vertices are adjacent and there s no backtracking (no subsequence $(\dots, u, v, u, \dots)$).

**Claim.** $|\text{Aut}(\text{Petersen})| = 120$.

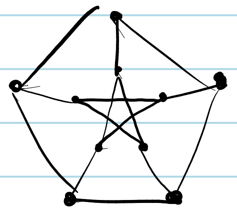

Fig 2

**Proof.** Consider the s-arcs up to $s = 3$. E.g. $(1,2,3,4)$ is a 3-arc. Claim $G_\alpha = \langle 1 \rangle$ for $\alpha$ any 3-arc. Then by Obstab $|G| = |G_\alpha||\alpha G| \leq 120$. ☐

| s | #s-arcs |
|---|---|
| 0 | 10 |
| 1 | 30 |
| 2 | 60 |
| 3 | 120 |

Table 3.

**Claim.** $\text{Aut}(J(7,3,1)) \cong \text{Sym}(8)$. **Pf.** Exer. (Text) ☐

## How Many Orbits Does G Have?

Suppose $G$ acts on $\Omega$. If $g \in G$, let $\text{fix}(g)$ be the set of points that are fixed by $G$.

**Lemma.** (Burnside) The number of orbits of $G$ is $\frac{1}{|G|} \sum_g |\text{fix}(g)|$ — the average size of the points fixed by elems $g \in G$.

**Remark.** Has a genfunc version called Pólya's Theorem.

Consider $\text{Sym}(n)$ acting on $K_n$. Then $\text{Sym}(n)$ acts on $E(K_n)$, and thus on subsets of $E(K_n)$ — otherwise known as subgraphs. The orbits of $\text{Sym}(n)$ on $E(K_n)$ are the isomorphism classes of graphs on $n$ vertices. ← $\boxed{\text{question 1}}$

# Automorphism Groups of Cayley Graphs

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Which finite groups appear as the automorphism group of a graph? **Thm.** (Frucht) All of them. (paraphrased)

The automorphism group of a "Cayley diagram" (Cayley digraph) for $G$ is $G$. This will be shown over the course of the full proof. ← directed graph

To take a 'diagram' and turn it into a graph (which makes it undirected), replace each directed edge with an asymmetric graph like 

Now which groups $G$ have a Cayley graph $X = X(G, C)$ such that $\mathrm{Aut}(X) \cong G$? (Such an $X$, if it exists, is called a graphical regular representation – abbreviates to GRR.)

Set $\Gamma = \mathrm{Aut}(X)$, where $X = X(G,C)$. Then $G \le \Gamma$. Suppose $\gamma \in \Gamma$. Then there exists $g \in G$ such that $1g = 1\gamma$. So $1\gamma g^{-1} = 1$ and thus $\gamma g^{-1} \in \Gamma_1$ ← stabilizer of 1 in $\Gamma$. Hence $\gamma \in \Gamma_1 g \subseteq \Gamma_1 G$ and $\Gamma_1 \cap G = \langle 1 \rangle$. It follows by Orb. Stab. that $|\Gamma| = |\Gamma_1||G|$.

The GRR question is equivalent to asking which groups $G$ admit Cayley graphs with trivial vertex stabilizers.

A permutation group $G$ on $\Omega$ is regular (sometimes simply transitive) if $G$ is transitive on $\Omega$ and $G_1 = \langle 1 \rangle$.

conjugation

$x^g = g^{-1}xg$

The map $x \mapsto x^g$ is an automorphism ~~that~~ fixes the identity of a group. Never transitive, but useful.

If $G$ admits an automorphism $\varphi$ so that $C^\varphi = C$ then $\varphi \in \mathrm{Aut}(X(G,C))_1$. It follows that should $G$ admit an automorphism fixing $C$, then $\mathrm{Aut}(X)_1 \ne \langle 1 \rangle$, and is not a GRR. For example, abelian $G$ with exponent greater than 2 has $\varphi(x) = -x$.

$\max_{g \in G} \{\mathrm{ord}(g)\}$

Any group admitting an automorphism mapping each element to itself or its inverse cannot have a GRR. In addition to abelian groups, there are so-called generalized dicyclic groups.

**Theorem.** If $G$ is not abelian or generalized dicyclic, has exponent greater than 2 and order at least 32, then $G$ has a GRR.

**Lemma.** Suppose $C$ generates $G$ and the subgraph of $X = X(G,C)$ induced by $C$ is asymmetric. Then $X$ is a GRR.

**Proof.** If the subgraph induced by $C$ is asymmetric and $\varphi \in \mathrm{Aut}(X)_1$, then $\varphi$ fixes each of the vertices in $C$. (See Figure 1.) Then induct on the distance from 1.

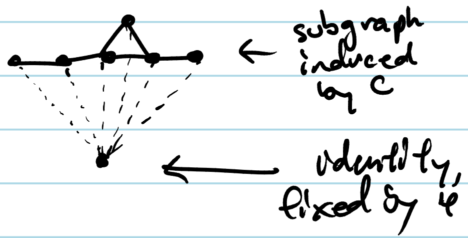

Fig. 1

## Cubic Graphs

**Claim.** There is no automorphism on a cubic graph fixing the identity of any order other than $2^m 3^n$! E.g. no factors of higher primes.

**Proof.** Cannot permute neighbours of a cubic vx in order 5. □

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# Cayley Graphs for Sym(n)

The symmetric group is generated by transpositions. What are minimal connecting sets of transpositions?

The transpositions in Sym(n) can be viewed as the edges of $K_n$.

Claim. A set of transpositions generates Sym(n) iff the graph is connected. Pf. Exer.

Hence the set of trees on n vertices generate Sym(n). Eg.

$$E_7 = \begin{array}{ccccccccccc} 1&-&2&-&3&-&4&-&5&-&6\\ &&&&&&|&&&&\\ &&&&&&7&&&& \end{array}$$ generates Sym(7), and $\mathrm{Aut}(E_7) = \langle 1 \rangle$.

A question: Is $X(\mathrm{Sym}(7), E(E_7))$ a GRR?

## Linear Cayley Graphs

Let $\mathbb{F} = GF(q)$. Let $V(d,q) = \mathbb{F}^d$. Let $M \in \mathrm{Mat}_{d\times n}(\mathbb{F})$ such that no two distinct columns are linearly dependent. $X(V(d,q), M)$ has two vertices x and y adjacent if $y-x$ is a non-zero scalar multiple of a column of M.

## Remarks

- $X(V(d,q), M)$ is Cayley (obv.)
- If $\mathbb{F} = \mathbb{Z}_2$ then $X(M)$ is cubelike, and all cubelikes arise in this fashion.
- If c is a column of M, $\{\lambda c \mid \lambda \in \mathbb{F}\}$ forms a clique of size q.
- The Hamming graph $H(d,q) = X(I_d)$. $H(d,2)$ is the d-cube. Two vertices are adjacent if their strings differ in 1 pos.
- Valency of $X(M)$ is $(q-1)m$, size is $q^d$.
- $X(M)$ is connected iff $\mathrm{rank}(M) = d$.
- If $Q \in GL(d,q)$ (Q is invertible $d\times d$ matrix) then $X(QM) \cong X(M)$.

Claim. If $\varphi \in \mathrm{Aut}(G)$ and $C \subseteq G$ ~~then~~ $X(C^\varphi) \cong X(C)$.

Remark. $M' = QM$ for invertible Q $\iff$ $\mathrm{Row}(M) = \mathrm{Row}(M')$

This means we can assume any matrix with some row space of dim d (rank $= d$) is in RREF i.e. upper-left $d\times d$ block is the identity.

- If rank $M = d$, $H(d,q)$ is a spanning subgraph of $X(M)$.
- If $a^T M$ has no zero entries, $X(M)$ is q-colourable. Note that $a^T M \in \mathrm{Row}(M)$. (Vxs in Col(M) though?)

To see this, assume rank $M = d$. Then $v \in V(X(M))$ lies in Col(M). So it is equal to $Mx$ for some x. Assign colour $a^T Mx$ to v. This is a colouring by affine hyperplanes.

# "Quotients" (Let's call it.)

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Consider $X = X(G,C)$ and $H \leq G$. One can look at the subgraph induced by $H$, but not much happens in terms of interesting structure.

However one can look at the right and left cosets of $H$. For instance, considering the right cosets $Hx$, we get a partition into cosets and representations and stuff I don't get. But they play nice. The left cosets suck, but they can be considered the orbits of $H$ in its action on $X$. In fact, each left coset induces Cayley subgraphs. If $H$ is normal, left and right cosets are equal and things get nice.

If $H \trianglelefteq G$, we can even construct a quotient graph. How? Read on.

We say a Cayley graph $X(G,C)$ is normal if $g^{-1}Cg = C$ $\forall g \in G$.

## Quotients of Linear Graphs

Let $X = X(M)$ be a linear Cayley graph. Partition $X$ into cosets of $H$. Assume any two vertices in $H$ are at distance 3 or greater. In this case there is a well-defined quotient graph, which is a Cayley graph for $G/H$.

## Coset Graphs in Vector Spaces.

Let $S \leq V(d,q)$. If $u,v \in V(d,q)$, the Hamming distance $h(u,v)$ is the number of coordinates $i$ where $u_i \neq v_i$. Equivalently in our case, it is the number of non-zero entries of $v-u$. The weight of $u$ is $h(u,0)$. So assume any two distinct vectors of $S$ have distance at least 3.

The coset graph of $S$ has the cosets of $S$ as its vertices. Two cosets $x+S$ and $y+S$ are adjacent if $y-x+S$ contains a vector of weight 1, i.e. a scalar multiple of a standard basis vector of $V(d,q)$.

$X(V(d,q), Id)$

Claim. The coset graph of $S$ is a quotient of the Hamming graph $H(d,q)$!

So we have that coset graphs are linear.

Choose the matrix $N$ such that $\ker N = S$. $Nv = 0 \Leftrightarrow v \in S$, etc. We also have that $u,v \in V(d,q)$ are in the same coset if $Nu = Nu$. The vectors in $Col(N)$ correspond to the cosets of $S$. Two vectors in $Col(N)$ correspond to adjacent cosets if their difference is a non-zero scalar multiple of $Ne_i$ for some $i$. The quotient is $X(N)$.

## More on normal Cayleys.

$X = X(G,C)$, $C = g^{-1}Cg$ $\forall g \in G$, i.e. $C$ is a union of conjugacy classes.

Claim. The maps $x \mapsto g^{-1}xg$ and $x \mapsto gx$ are automorphisms of $X$.

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# Embeddings (in Cayley Graphs)

Given a graph $Y$, is there a Cayley graph for $G$ such that $Y$ is an induced subgraph?

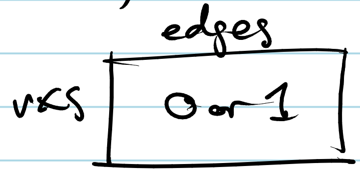
Begin by assigning elements of the group $g_1, g_2, \dots, g_n$ to the vertices $1, 2, \dots, n$ of $Y$. Define $C = \{g_j g_i^{-1} \mid i \sim j \text{ in } Y\}$, and set $X = X(G, C)$. So when is the subgraph of $X$ induced by $g_1, \dots, g_n$ isomorphic to $Y$?

We require the following conditions: that the map $i \mapsto g_i$ is injective, and that if $i \not\sim j$, then $g_j g_i^{-1} \notin C$. This is sufficient, but doesn't make things very easy.

For example, taking any $Y$ and $G = \mathbb{Z}_2^{|V(Y)|}$ use $i \mapsto e_i$ and $C = \{e_j - e_i \mid i \sim j \text{ in } Y\}$, and now $Y$ is an induced subgraph of $X(G, C)$. Such a thing can also be done with circulants and other similarly generic classes of Cayley graphs, but the interesting point is that this $X(G, C)$ is a linear Cayley.

$X(G, C)$ above can be written $X(M)$ for some $M$ — in fact, this $M$ is the incidence matrix of $Y$. (Figure 1)

Fig. 1



## Homomorphisms of Graphs

A homomorphism of graphs from $X$ to $Y$ is a map $V(X) \to V(Y)$ preserving adjacency. Notably, adjacent vertices cannot be mapped to the same vertex unless one allows loops, which unfortunately trivializes the affair entirely — take the "constant" map to the looped vertex. For example, if $X \leq Y$, then $X \to Y$.

Use $X \to Y$ as shorthand for "there exists a homomorphism mapping $X$ to $Y$".

If $f : X \to Y$ is a homomorphism and $y \in Y$, then

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}$$

is called the fibre of $f$ at $y$. The fibres partition $X$ if $f$ is surjective, and this partition is the kernel of $f$. Notice every fibre is a coclique. Consequently, a graph $X$ is $m$-colourable iff $X \to K_m$.

$X$ and $Y$ are homomorphically equivalent if $X \to Y$ and $Y \to X$. Write $X \leftrightarrow Y$. For example, if $X$ is bipartite and nonempty, $X \leftrightarrow K_2$. Also, if $X \leq Y$ and $Y \to X$, then $X \leftrightarrow Y$.

Observe that $\to$ as a relation is reflexive ($X \to X$) and transitive ($X \to Y$ and $Y \to Z$ implies $X \to Z$). Is $\to$ an order? If $X \leftrightarrow Y$ it does not follow that $X = Y$, so $\to$ is a preorder. $\leftrightarrow$ is an equivalence relation, then, and thus $\to$ is a partial order on homomorphism classes. Use poset ideas, i.e. meets, joins, lattices, etc.

We denote the greatest lower bound (if it exists) of $a$ and $b$ in a poset by $a \wedge b$ (meet) and similarly the least upper bound $a \vee b$ (join).

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# The Lattice of Homomorphic-Equivalence Classes

**Theorem.** The poset of hom-equivalence classes of graphs is a lattice.

**Proof.** We show that each pair of elements of the poset has a join and a meet. This is essentially a category-theoretic problem (direct and inverse limits). For graphs $X$ and $Y$,

join $\vee$ least upper bound

meet $\wedge$ greatest lower bound.

$$\begin{tikzcd} & Z & \\ & X \vee Y \arrow[u, dashed] & \\ X \arrow[ur] \arrow[uur, bend left] & & Y \arrow[ul] \arrow[uul, bend right] \end{tikzcd}$$

← join

meet →

$$\begin{tikzcd} X & & Y \\ & X \wedge Y \arrow[ul] \arrow[ur] & \\ & Z \arrow[u, dashed] \arrow[uul, bend left] \arrow[uur, bend right] & \end{tikzcd}$$

We take $X \vee Y$ to be the disjoint union $X \sqcup Y$ and $X \wedge Y$ to be the graph product $X \times Y$, defined as follows:

$V(X \times Y) = V(X) \times V(Y)$ and $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1 \sim x_2$ and $y_1 \sim y_2$.

Observe that the diagonal $\{(x,x) \mid x \in V(X)\}$ of $X \times X$ induces a subgraph isomorphic to $X$, so $X \to X \times X$. Also see that the projection maps $p_X$ and $p_Y$ defined by $(x,y) \mapsto x$ and $(x,y) \mapsto y$ respectively are homomorphisms from $X \times Y$ into $X$ and $Y$. So $X \times Y \to X$ and $X \times Y \to Y$. So consequently $X \times X \leftrightarrow X$.

It remains to prove $X \times Y$ is in fact the greatest lower bound of $X$ and $Y$. For any $Z \to X, Y$, with maps $f: Z \to X$ and $g: Z \to Y$, take $h: Z \to X \times Y$ by $h(z) = (f(z), g(z))$, which is a homomorphism. So $X \wedge Y = X \times Y$. ▨

So now consider $X \to Y$, where $Y \not\to X$. Can you squeeze in a $Z$ such that $X \to Z \to Y$? That is, is this lattice dense? It is difficult to prove, but yes. (See text.)

## Cores

Another thing to consider: in analogy with e.g. the integers mod $n$, is there a nice choice of "smallest" representative for each hom-equiv. class of graphs?

**Prop.** The graphs homomorphically equivalent to $X$ with the minimum number of vertices are all isomorphic.

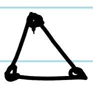
**Pf.** Suppose $Y_1$ and $Y_2$ are two such graphs, with $Y_1 \xrightarrow{f} X \xrightarrow{g} Y_2$. Then $g \circ f$ is a hom from $Y_1$ to $Y_2$. It will be an isomorphism (Exer.) ▨

A minimum graph hom-equiv. to $X$ is called a core of $X$, denoted $X^\bullet$. If the graph $Y$ is a core, $\mathrm{End}(Y) = \mathrm{Aut}(Y)$.

**Prop.** $X^\circ$ is isomorphic to an induced subgraph of $X$. **Pf.** Same trick as before. $X^\circ \xrightarrow{f} X \xrightarrow{g} X^\circ$, so $h = g \circ f$ is surjective, and so $f$ is injective. (Exer. Show $X^\circ$ is induced.) ▨

Some examples of cores:

↖ any non-empty bipartite graph.



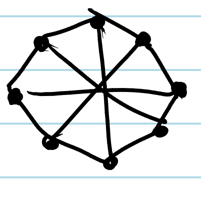
← not hom-equiv to $K_2$.

$K_n$

core for all $n$

Petersen

↖ is it a core?



← mobius ladder? core?

??? how do you even tell ???

So this core-finding problem is hard in the general case. What about vertex-transitive graphs? Cayley graphs? Read on...

A graph $Y$ is a retract of $X$ if there is a retraction from $X$ to $Y$, i.e. a homomorphism $f: X \to Y$ such that $f \restriction Y = \mathrm{id}_Y$. There is a retract from $X$ to $X^\circ$, and in general homomorphisms from $X$ to $X^\circ$ can be "unscrewed" into retractions by composing them with things in $\mathrm{Aut}(X^\circ)$ in a special way.

$f \restriction Y$ – $f$ restricted to $Y$.

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# Foldings

A *simple folding* is an identification of two vertices at distance 2. A *folding* is a composition of simple foldings.

**Lemma.** A retraction is a folding.

**Proof.** Given a retract $f: X \to Y$, partition $X$ into fibres. Choose $y \in Y$ and then $a \notin Y$ s.t. $a \sim y$. Then $f(a) \sim y$ and $f(a) \in Y$. Identify $a$ and $f(a)$ to get a simple folding, and the trick is to then argue that the remainder of the action of $f$ is still a retraction. (Exer.) ▨

A homomorphism $f$ is a *local injection* if its restriction to the neighbours of a vertex is injective.

**Theorem.** If $h: X \to Y$ is a homomorphism then it can be factored $h = f \circ g$ where $f$ is a local injection and $g$ is a folding.

**Proof.** In the text. The key step is to define a relation on $\ker h$: $u \approx v$ iff $f(u) = f(v)$ and either $u = v$ or $\mathrm{dist}(u,v) = 2$. If $\simeq$ is the transitive closure of $\approx$, its equivalence classes form a partition that refines $\ker f$. ← prob $h$? ▨

A graph $X$ *covers* a graph $Y$ if there is a homomorphism $f: X \to Y$ such that $f$ is a local isomorphism. ← ???

If the neighborhoods of the vertices of $X$ are cores, then $X$ is a core. This doesn't help much for determining if triangle-free graphs are cores, but then you know e.g. 5-cycles map to 5-cycles.

## Cores of Vertex-Transitive Graphs

**Lemma.** Suppose $G$ acts transitively on $X$ and $|G_1| = m$. If $X^{(m)}$ is the graph attained by multiplying each vertex by $m$ (replacing each vertex with an $m$-coclique and connecting all possible edges between "adjacent" cocliques), then it is a Cayley graph for $G$ and $X$ is a retract of $X^{(m)}$.

**Proof.** Define $C = \{g \in G \mid 1g \sim 1\}$. Then $C = C^{-1}$. It is a union of right cosets of $G$ and $C \cap G_1 = \emptyset$. Let $Y = X(G, C)$ and claim $Y \cong X^{(m)}$. ▨

**Lemma.** The core of a vertex-transitive graph is vertex-transitive.

**Proof.** Let $X$ be vertex-transitive and let $f$ be the retract $X \to X^\bullet$. Take $u, v \in X^\bullet$ and $\gamma \in \mathrm{Aut}(X)$ such that $u\gamma = v$. Then $f \circ \gamma$ maps $X^\bullet \to X^\bullet$ and $u \mapsto v$. Hence $X^\bullet$ is vertex-transitive. ▨

**Lemma.** If $X$ is vertex-transitive then $|V(X^\bullet)| \mid |V(X)|$. **Pf.** Next time!

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## More on Cores

**Theorem.** If $X$ is vertex-transitive, $|V(X^\bullet)|\,\big|\,|V(X)|$.

**Proof.** Consider $f : X \longrightarrow X^\bullet$. $f$ induces a partition on $X$ by its fibres. (Call it $\ker f$.) If $g \in \mathrm{Aut}(X)$, $f \circ g$ must be a surjection, since its codomain is a core. $(X^\bullet)^g$ is "skewed" w.r.t. $\ker f$. Now, the crux of the lemma—

$$|\mathrm{Aut}(X)| = |\text{fibre of } f| \cdot \#\text{ translates of } (X^\bullet)^g \text{ on a given vx.}$$

Consider automorphisms, not "copies of $X^\bullet$". Exer. Make sense of this proof. ☒

## Free Cayley Graphs

Let $Y$ be a graph and $p$ be a prime. Then $\mathbb{Z}_p(Y) = X(M)$ where $M$ is the incidence matrix with columns $e_i - e_j$ when $ij \in E(Y)$. $\mathbb{Z}_p(Y)$ is the free Cayley graph for $Y$ over $\mathbb{Z}_p$. The vertices $e_1, \dots, e_n$ in $\mathbb{Z}_p(Y)$ induce a copy of $Y$.

For example, taking $\mathbb{Z}_2(K_3)$: $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, so $|V(\mathbb{Z}_2(K_3))| = 8$ and in fact $\mathbb{Z}_2(K_3)$ is isomorphic to $2K_4$: it is disconnected.

**Theorem.** Suppose $Z$ is a Cayley graph on $\mathbb{Z}_p^d$ and there is a homomorphism $f : Y \to Z$, for some graph $Y$. Then there is a homomorphism $\hat{f} : \mathbb{Z}_p(Y) \to Z$ such that $\hat{f}$ is linear and the following diagram commutes:

$$\begin{tikzcd} & Y \arrow[r, "f"] \arrow[d, "\beta"'] & Z \\ \text{From the text} & \mathbb{Z}_p(Y) \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

**Proof.** The input is $Y \xrightarrow{f} Z$ and $\mathbb{Z}_p(Y)$, so we must construct $\hat{f}$. It can be done by extending the map on the basis $e_1, \dots, e_n$ to a map on the whole vector space (this is a theorem of linear algebra), so it remains to check that $\hat{f}$ is a graph homomorphism.

component containing 0

Suppose $u \sim v$ in $\mathbb{Z}_p(Y)_0$. Then $v - u = \lambda(e_i - e_j)$ for some $i \sim j$ in $Y$ and $\lambda \neq 0$.

$$\hat{f}(v) - \hat{f}(u) = \hat{f}(v-u) = \lambda \hat{f}(e_i - e_j) = \lambda \hat{f}(e_i) - \lambda \hat{f}(e_j) = \lambda f(i) - \lambda f(j)$$ and we're done. ☒

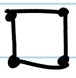
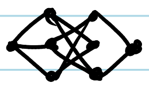
**Theorem.** (Payan) No cubelike graph has $\chi = 3$. ☒, but the proof would proceed something like, taking $Z$ cubelike and not bipartite, observe

if $C_{2k+1} \to Z$, then
$$\begin{tikzcd} C_{2k+1} \arrow[r] \arrow[d, "\beta"'] & Z \\ \mathbb{Z}_2(C_{2k+1}) \arrow[ur] & \end{tikzcd}$$
and $\chi(\mathbb{Z}_2(C_{2k+1})) = 4$, so $\chi(Z) \geq 4$.

Another bounty of neat results comes from
$$\begin{tikzcd} K_4 \arrow[r] \arrow[d] & Z \\ \mathbb{Z}_3(K_4) \arrow[ur] & \end{tikzcd}$$
; $\chi(\mathbb{Z}_3(K_4)) = 7$!

## Folded Cubes

We construct a graph from $Q_d$ as follows. Let $\pi$ be a partition of antipodal pairs in $Q_d$ ($\mathrm{dist}(x,y) = d$). The folded cube $Q_d/2$ has the antipodal pairs as vertices, with an edge adjacent to another if one vertex in one pair is adjacent to either in the other pair.

| $d$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $Q_d$ |  |  | $Q_4$ | $Q_5$ |
| $Q_d/2$ |  |  | $K_{4,4}$ | Clebsch |

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28 Jan 2018

# Halved and Folded Cubes

We defined the folded d-cube as a quotient of $Q_d$. It can also be obtained by adjoining antipodal pairs of $Q_{d-1}$.

Observe that $Q_d/2 \cong \mathbb{Z}_2(C_d)$. Something something assignment question. (???)

If $C_{2k+1} \to Z$ for $Z$ cubelike then $\mathbb{Z}_2(C_{2k+1}) \to Z$. But $\chi(\mathbb{Z}_2(C_{2k+1})) = 4$. (Take $Z = K_n$, and not 3-colourable because it contains a "generalized Mycielski graph".)

This is decidedly non-trivial business!

### Mycielski graphs

- triangle-free - certain $\chi$ val
- put a cone on $L_n \times X$
- $L_n =$ 

- $L_n \times X$ is like taking $P_n \times X$ (n copies of cocliques) except make the last a copy of X
- put the cone on the other end (not the loopy end)
- $\chi = 4$ - induced subgraph of $Q_d/2$

## More on Halved Cubes

The halved d-cube $\frac{1}{2}Q_d$ has as vertex set one of the parts in a bipartition of $Q_d$, and vertices are adjacent in $\frac{1}{2}Q_d$ if they are at distance 2 in $Q_d$.

Observe $\frac{1}{2}Q_d = \mathbb{Z}_2(K_d)$. It can also be seen as the "distance 1 or 2" graph of $Q_{d-1}$. The neighbourhood of a vertex in $\frac{1}{2}Q_d$ is a line graph of the complete graph.

### line graph

$V(L(G)) = E(G)$
adjacent iff sharing exactly one vertex

For some examples, $\frac{1}{2}Q_4 = \mathbb{Z}_2(K_4) = \overline{4K_2}$, and $\frac{1}{2}Q_5 = \overline{\text{Clebsh}}$. Note that because of this, cubelikes containing 5-cliques have $\chi \geq 8$. (!!)

## Distance-Transitive Graphs

These graphs have "the maximum amount of symmetry possible". A graph is distance-transitive if there is an automorphism mapping any pair of vertices at distance d to any other such pair, for all d ≤ diameter.

Some examples: $K_n$, $K_{n,n}$, $L(K_n)$, $L(K_{n,n})$, $C_n$, Petersen, L(Pete).

If a distance-transitive graph is not a core, its core is a complete graph. This property is called core-completeness. (Actually let's prove that)

**Theorem.** A distance-transitive graph is core-complete.

**Lemma.** If X is connected and regular, and Aut(X) acts transitively on pairs of vertices at distance two, then X is core-complete.

**Proof.** Suppose $X^\circ \neq X$ is not complete. Then there exist $a \nsim b$ in $X^\circ$. There is also a $y \in X^\circ$, $y \sim z \in X \setminus X^\circ$, $y \sim f(z) = z' \in X^\circ$. Take $\gamma : (a,b) \mapsto (z,z')$ and compose with f on left for a contradiction.

**Pf. of Thm.** Exercise.

Fun observation. Take X vertex-transitive on $2^k$ vertices, and suppose X has a triangle. That triangle cannot be a core, so X is not 3-colourable.

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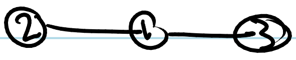
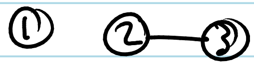
# Adjacency Matrices

For a graph $X$, create an adjacency matrix $A$ whose rows and columns are indexed by $V(X)$ and take $A_{ij} = [i \sim j \text{ in } X]$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Properties:

$-A = A^T$ $\quad - A_{ii} = 0$ $\quad - A(X)$ and $A(\bar{X})$ are not very similar

Recall $J$ is the matrix of all ones and $I$ is the identity.

The adjacency matrix of the complement of a graph is $J - I - A$.

Denote by $\Delta$ the diagonal matrix whose i-th diagonal entry is $\deg_X i$.

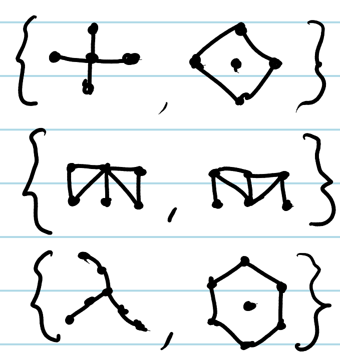
The Laplacian of a graph is $\Delta - A$. The normalized Laplacian is $\Delta^{-\frac{1}{2}} A \Delta^{-\frac{1}{2}}$.

Two graphs $X, Y$ are isomorphic if there is a permutation matrix $P$ such that

$$P^T A(X) P = A(Y).$$

So $\mathrm{Aut}(X)$ can be identified with all perm. matrices commuting with $A$.

Notice that the characteristic polynomial is an invariant. Does it entirely determine the graph? The answer is no. ← some counterexamples



Denote the characteristic polynomial of a graph by

$$\phi(X,t) = \det(tI - A) = t^n + a_1 t^{n-1} + \dots + a_{n-1} t + a_n.$$

It is monic with degree $|V(X)|$. Notice $a_1 = 0$ because it is the trace, and $a_n = \det(-A) = (-1)^n \det A$.

**Theorem.** (Laplace) $a_r$ is the sum of the principal $r \times r$ minors of $A$.

So $a_1 = 0$ and $a_2 = -|E(X)|$. $a_3$ is related to triangles?

Define the shear product $M \circ N$ by $(M \circ N)_{ij} = M_{ij} N_{ij}$.

**Claim.** $\det A = \sum_{P \in \mathrm{Sym}(n)} \det A \circ P = \sum_{P \in \mathrm{Sym}(n)} \mathrm{sign}(P) \mathrm{wt}(A \circ P)$

## Eigenthings of Adjacency Matrices

Eigenvectors are functions on $V(X)$. If $f \in \mathbb{R}^{V(X)}$ then $(Af)(u) = \sum_{v \sim u} f(v)$.

For instance, $A(K_n)\underline{1} = (n-1)\underline{1}$. In general $\underline{1}$ is an eigenvector of regular graphs.

Note that you can get orthogonal bases of eigenvectors, and being orthogonal to the $\underline{1}$ vector means that the entries sum to zero.

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# Eigenvectors

f is an eigenvector iff $(Af)(u) = \sum_{v \sim u} f(v)$ for all u.

Consider another usage: take the dodecahedron with unit vectors for vertices. Applying the adjacency matrix is like adding the neighbouring unit vectors. This gives a linear embedding of the dodecahedron into $\mathbb{R}^3$. (Why?)

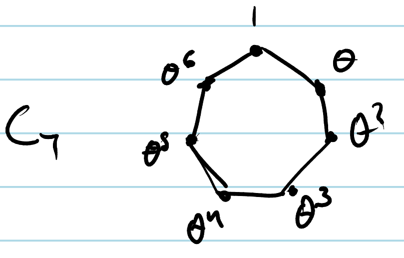
## Complete Bipartite Graphs

Consider $K_{n,n}$. Nonzero constant vectors are eigenvectors with eigenvalue n. Suppose also f sums to zero on each colour class. Then this is an e/vec with e/val 0. These e/vecs span a subspace of $\mathbb{R}^{2n}$ of dimension $2n-2$. (Must be less than $2n-1$.) The orthogonal complement to the sum of these two subspaces is spanned by the vector +1 on one colourclass and $-1$ on the other, eigenvalue $-n$.

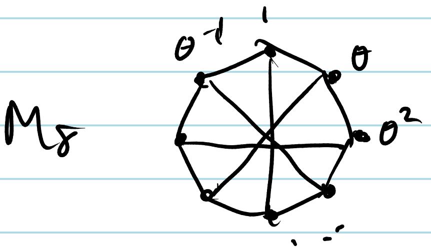
Why is the dimension of the 0 e/space $2n-2$? "Rank plus nullity theorem." Define the linear map $f \mapsto$ (sum on col. class 1, sum on col. class 2) $\longleftarrow$ ?

Consider now $K_{m,n}$. The $K_{n,n}$ arguments won't necessarily work, as they only give you rational e/vals, and e.g. the spectrum of $K_{1,2}$ is $\sqrt{2}, 0, -\sqrt{2}$. The functions summing to zero on each colour class are e/vecs w/ e/val 0, a subspace whose dimension is also $m+n-2$. The orthogonal complement of this subspace consists of vectors constant on colour classes. Normalizing one class to 1 and solving, we see the spectrum is $\sqrt{mn}^{(1)}, -\sqrt{mn}^{(1)}, 0^{(m+n-2)}$.

## Cycles (and Circulants?)



Regular so constant vec is eigen. Consider $C_7$. Take $\theta^7 = 1$ a seventh root of unity. Assigning $\theta^i$ to the vx i we get an eigenvalue $\theta + \theta^{-1}$. This is for all 7 roots of unity, so we have some good e/vecs. Using $\theta = 1$ gives the expected constant e/vec of regularity.



Trying this for the Möbius ladder gives an e/val of $\theta + \theta^4 + \theta^{-1}$.

**Theorem.** If $X = X(\mathbb{Z}_n, C)$ is a circulant, then the eigenvalues of X are $\sum_{r \in C} \theta^r$ for each n-th root of unity $\theta$. $\square$

Notice that the matrix obtained by writing all the eigenvectors together is a Vandermonde matrix.

## Cayley Graphs for Abelian Groups.

A character of a group G is a homomorphism $G \to \mathbb{C}^*$ the multiplicative group of complex numbers. If G is finite then this goes into the unit circle. For example, $\mathbb{Z}_n \to$ n-th roots of unity; $\mathbb{Z}_2^d \to \mathbb{C}^*$ by $\psi_a(u) = (-1)^{a^T u}$.

If $\chi$ is a character of G and $S \subseteq G$, $\chi(S) = \sum_{g \in S} \chi(g)$. (Define.)

**Thm.** If G abelian and $X = (G, C)$ and $\psi$ is a character for G, $\psi$ is an e/vec for X with e/val $\psi(C)$.

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# Characters

If $G$ is abelian and $\varphi$ is a character on $G$, then $\varphi$ is an eigenvalue for $X(G,C)$ with eigenvalue $\varphi(C)$.

If $\psi$ is a character, then so is ~~its~~ complex conjugate $\bar{\psi}$, and the product $\varphi\psi$. Notice that $\varphi\bar{\varphi}=1$ is the trivial character. Any power of a character is also a character.

It follows that characters form a group, the character group. If the group is cyclic, the character group is isomorphic to the group.

If $f$ and $g$ are complex functions on $G$, then $\langle f,g\rangle = \sum_{x\in G}\overline{f(x)}g(x)$. This is the complex inner product on the space of complex functions on $G$. If $\varphi$ and $\psi$ are characters, $\langle\varphi,\psi\rangle = (\varphi^{-1}\psi)(G)$.

**Lemma.** If $H\leq G$ and $\psi$ is a $G$-character, then either $\psi(H)=0$ or $\psi(h)=1$ for all $h\in H$.

**Proof.** If $h\in H$ and $x\in G$ then $\psi(hx)=\psi(h)\psi(x)$. Since $hH=H$,

$$\psi(H)=\psi(hH)=\sum_{x\in G}\psi(hx)=\sum_{x\in G}\psi(h)\psi(x)=\psi(h)\sum_{x\in G}\psi(x)=\psi(h)\psi(H).$$

So either each $h$ has $\psi(h)=1$, or $\psi(H)=0$. ■

Suppose $G$ and $H$ are abelian groups and $\psi$ (resp. $\varphi$) is a character on $G$ (resp. $H$), then the map $(g,h)\mapsto\psi(g)\varphi(h)$ in $G\times H\to\mathbb{C}$ is a character.

## Cubelike Eigenstuff

If $a\in\mathbb{Z}_2^d$ then the map $\theta_a$ sending $u\mapsto(-1)^{a^Tu}$ is a character. This corresponds characters to elements of $\mathrm{Row}(M)$ w.r.t. $X(M)$. Tune in next time for the epic exciting conclusion.

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# Cubelike Eigenstuff II: Characteristic Boogaloo

Take $M \in \mathbb{Z}_2^{d \times m}$, consider the eigenvalues of $X(M)$. We have characters $\tau_a : u \mapsto (-1)^{a^T u}$.

The eigenvalue belonging to a character $\tau_a$ is $\tau_a(C)$, where here $C = \{Me_i \mid i \leq m\}$. So

$$\tau_a(C) = \sum_{i=1}^{m} (-1)^{a^T M e_i} = wt(a^T M)(-1) + m - wt(a^T M) = m - 2wt(a^T M).$$

$a^T M$ is a code word (as in binary codes) in $Row(M)$. If $a_r$ is the number of words of weight $r$ in a code $C$, the "weight enumerator" is

$$W_C(x, y) = \sum_{r=0}^{m} a_r x^r y^{m-r}$$

Hence the eigenvalues of $X(M)$ are the integers $m - 2r$, where $r$ runs over the weights of the vectors of $Row(M)$, and the multiplicity of $r$ is $a_r$.

## Cayley Graphs for $\mathbb{Z}_3^d$

Take the character $\tau_a : u \mapsto \omega^{a^T u}$ where $\omega^3 = 1$ ($\omega$ is a cube root of unity.) The eigenvalues of $X(M)$ are $\tau_a(C) = \sum_{i=1}^{m} (\omega^{a^T M e_i} + \omega^{-a^T u M e_i}) = 2m - 3wt(a^T M) \in [-m, 2m]$

## Bounds on Eigenvalues

Something something Perron-Frobenius Theorem.

**Claim.** Suppose $Az = \lambda z$, $z \neq 0$, $\lambda \geq 0$. If $X$ is connected, $z > 0$. (Exer.)

**Claim.** If $X$ connected, $Az = \lambda z$, $z > 0$ then $\lambda$ is simple.

**Pf.** Suppose $Aw = \lambda w$ and $z, w$ lin. indep. Then $\exists v \in span\{z, w\}$ with $v \geq 0$ and some 0 entry. ■

If $Az = \lambda z$ then $|Az| = |\lambda||z|$; $|(Az)_i| = |\sum_j A_{ij} z_j| \leq \sum_{j \sim i} |z_j|$, so $A|z| \geq |Az| = |\lambda||z|$.

"subharmonic vectors".

Suppose $y^T A = \theta y^T$. Then $|\theta||y| \leq A|y|$. If $Az = \rho z$ for $z > 0$ then

$|\theta| \cdot z^T |y| \leq z^T A |y| = \rho |z|^T |y|$ and $|\theta| \leq \rho$ ← spectral radius.

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# (More) Graph Eigenvalues

Take spectral radius $\rho$ and $\hat{\Delta}$ average degree.

**Lemma** Assume $X$ is connected. Then $\rho \geq \hat{\Delta}$, equality iff $X$ is regular

Consider the optimization problem $\max_{\|x\|=1} x^TAx$. Assume $x$ maximizes and look at $x+h$ for small $h$ that have $x^Th=0$.

$$(x+h)^T(x+h) = x^Tx + h^Tx + x^Th + \cancel{h^Th} \overset{h \text{ is small}}{\approx} x^Tx$$

$(x+h)^TA(x+h) = x^TAx + x^TAh + h^TAx + \cancel{h^TAh} = x^TAx + 2x^TAh$. Since $x$ is a local max, $x^TAh$ must be zero. So if $x^Th=0$ then $x^TAh=0$, i.e. if $h \in x^\perp$ then $h \in (Ax)^\perp$. This means $x^\perp \subseteq (Ax)^\perp$. (Exer. $x^\perp = (Ax)^\perp$) Then $Ax = \lambda x$ for some $\lambda$.

**Corollary.** If $\rho \geq x^TAx$, $\|x\|=1$, take $x = \frac{1}{\sqrt{n}}\underline{1}$. Then $x^TAx = \frac{1}{n}\underline{1}A\underline{1} = \frac{2|E(X)|}{n} = \hat{\Delta}$.

## Spectral Decomposition

**Theorem.** Suppose $A$ is a symmetric $n \times n$ matrix with distinct eigenvalues $\theta_1 \geq \dots \geq \theta_m$ Then there exist matrices $E_1, \dots, E_m$ such that:

- $A = \theta_1E_1 + \dots + \theta_mE_m$
- $\sum_i^m E_i = I$
- $E_i = E_i^T$
- $E_iE_j = [i=j]E_i$.

The $E_i$ are symmetric idempotents, where the product of distinct ones is 0. These are unfortunately named orthogonal idempotents.

**Proof.** Suppose $D$ is an $n \times n$ diagonal matrix with distinct e/vals $\delta_1, \dots, \delta_m$. Then

$$D = \delta_1F_1 + \dots + \delta_mF_m, \text{ where } F_i \text{ is } [D = \delta_iI] \text{ pointwise on the diag.}$$

Then lift this to general matrices by taking unitary matrices for similarity. ∎

**Claim** $E_r$ represents orthogonal projection onto the $\theta_r$-eigenspace.

**Pf.** If $u \in \mathbb{R}^n$ then $AE_ru = \delta_rE_ru$ so $E_ru$ is in the $\theta_r$ e/sp. Then see that

$$\dim(\operatorname{im}(E_r)) = \operatorname{rank} E_r = \operatorname{mult}(\theta_r),$$

so $E_r$ is onto. Then show $x - Ex \perp Eu$ $\forall u$. (Note $(I-E_r)E_r = E_r - E_r = 0$.) ∎

### Some Review

Suppose $u_1, \dots, u_d$ is an orthogonal basis for a subspace $U$. If $y \in \mathbb{R}^n$, projection $\hat{y} \in U$ is $\hat{y} = \frac{\langle y, u_1\rangle}{\langle u_1, u_1\rangle}u_1 + \dots + \frac{\langle y, u_d\rangle}{\langle u_d, u_d\rangle}u_d$. So the matrix for this is $M = \sum_r^d \frac{u_ru_r^T}{u_r^Tu_r}$.

"Spectral Decomp Refinement" — Breaking up idempotents of nondistinct e/vals into idempotents constructed as above for each e/vec.

If $A = \sum_r \theta_rE_r$, $A^k = \sum_r \theta_r^kE_r$, $p(A) = \sum p(\theta_r)E_r$. So if $p_r(t) = \prod_{s \neq r} \frac{t-\theta_s}{\theta_r - \theta_s}$ then

$p_r(A) = E_r$ — the idempotents are polys in $A$!

# Walks, I guess...

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$(A^k)_{ij}$ is the number of walks from i to j in X of length k.

We have a generating function $\sum_k (A^k)_{ij} t^k$, which is just the entries of the function $\sum_k A^k t^k = (I-tA)^{-1}$. Call this $W(X,t)$, and the $(i,j)$ entry $W_{ij}(X,t)$.

$$(tI-A)^{-1}_{ii} = \frac{\phi(X\setminus i,t)}{\phi(X,t)}, \text{ so } (I-tA)^{-1}_{ii} = \frac{\phi(X\setminus i,t^{-1})}{\phi(X,t^{-1})}\frac{t^m}{t^m}t^{-1} = \frac{\hat{\phi}(X\setminus i,t)}{\phi(x,t)} \text{ ?}$$

Two graphs X and Y are *cospectral* if their adjacency matrices are similar. A vertex i in X is *cospectral* to j in Y if $\phi(X,t) = \phi(Y,t)$ and $\phi(X\setminus i,t) = \phi(Y\setminus j,t)$.

$$(tI-A)^{-1} = \sum_r \frac{E_r}{t-\theta_r} \Rightarrow (tI-A)^{-1}_{ii} = \sum_r \frac{(E_r)_{ii}}{t-\theta_r}$$

$$\sum_r E_r = I \Rightarrow \sum_r (E_r)_{ii} = 1$$

$$(E_r)_{ii} = e_i^T E_r e_i = e_i^T E_r^T E_r e_i = \|E_r e_i\|^2 \geq 0$$

The idempotents $E_r$ are positive semidefinite.

A matrix M is *positive semidefinite* (psd) if $M = M^T$ and $x^T M x \geq 0$ for all x. Write $M \succeq 0$. TFAE:

(a) $M \succeq 0$ (b) $x^T M x \geq 0 \ \forall x$ (c) $M = NN^T \ \exists N$ (d) e'vals of $M \geq 0$

Some remarks:

- principal submatrices of psd's are psd
- if $M \succeq 0$ and $M_{ii} = 0$ then all entries in i-th row and col are 0.
- if $M, N \succeq 0$, then so are $M+N$, $M \circ N$, $M \otimes N$
- if M psd and equality only when $x = 0$, M is pos.def. ($M \succ 0$)
- $M \succ 0 \iff M \succeq 0$ and invertible

*Hadamard product*

$(M \circ N)_{ij} = M_{ij} N_{ij}$

"pointwise"

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# Line Graphs

Let $B$ be the incidence matrix of a graph.

$B^TB = 2I + L$ ← adjacency matrix of line graph

→ e/vals of $2I+L$ are nonneg, so e/vals of $L$ are $\geq -2$

**Theorem.** If $AB$ and $BA$ are both defined, then they both have the same nonzero eigenvalues with the same multiplicities. (General $A$, $B$)

**Proof.** (Magic.)

$$\underset{\text{whatever}}{\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}\begin{bmatrix} I & A \\ B & I \end{bmatrix}} = \underset{\det(I-BA)}{\begin{bmatrix} I & 0 \\ 0 & I-BA \end{bmatrix}}, \quad \begin{bmatrix} I & A \\ B & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} = \underset{\det(I-AB)}{\begin{bmatrix} I-AB & A \\ 0 & I \end{bmatrix}}$$

So then $\det(I-AB) = \det(I-BA)$. Letting $A \to tA$, $AB$ and $BA$ have the same characteristic poly. □

We have $B^TB = 2I + L$. Also, $BB^T = \Delta + A$. So they have the same eigenvalues with the same multiplicities. If $X$ is $k$-regular, $\Delta = kI$ so the e/vals of $L$ (aside from $-2$) are of the form $\theta + k - 2$ for $\theta$ the e/vals of $X$. For example,

spectrum of $K_n = n-1^{(1)} \quad -1^{(n-1)}$

of $L(K_n) = 2n-4^{(1)} \quad n-4^{(n-1)} \quad -2^{(\binom{n}{2}-n)}$

spectrum of $K_{n,n} = n^{(1)} \quad 0^{(2n-2)} \quad -n^{(1)}$

of $L(K_{n,n}) = 2n-2^{(1)} \quad n-2^{(2n-2)} \quad -2^{(1+?)}$

If $X$ is regular with e/vals $k$ and $\{\theta_i\}$ then the e/vals of $\overline{X}$ are $n-1-k$ and $\{-\theta_i - 1\}$. (See that $A(\overline{X}) = J - I - A(X)$.)

Petersen $= \overline{L(K_5)}$ → spec of $K_5 = 4^{(1)} \quad -1^{(4)}$

$L(K_5) = 6^{(1)} \quad 1^{(4)} \quad -2^{(5)}$

$\overline{L(K_5)} = 3^{(1)} \quad -2^{(4)} \quad 1^{(5)}$

## Interlacing

If $a \in V(X)$, between each pair of e/vals of $X$, there is an e/val of $X \backslash a$.

e.g. spec of $P_3 = -\sqrt{2}^{(1)} \quad 0^{(1)} \quad \sqrt{2}^{(1)}$ and spec of $P_2 = -1^{(1)} \quad 1^{(1)}$

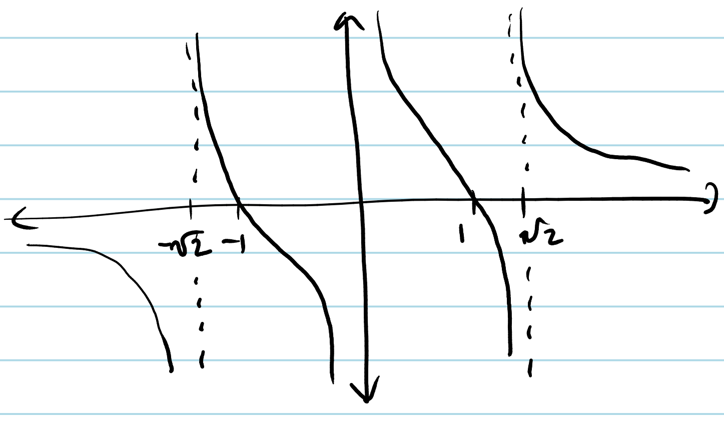
**Theorem.** Let $A$ be a Hermitian $n \times n$ matrix and $B$ be a principal $m \times m$ submatrix. Then $\theta_r(B) \leq \theta_r(A)$ and $\theta_r(-B) \leq \theta_r(-A)$. (Exer. use opt. trick)

**Claim.** Interlacing is equivalent to the statement that the residues at the poles of $\phi(X \backslash a, t)/\phi(X, t)$ are positive.

**Pf.**

$$\frac{d}{dt} \frac{\phi(X \backslash a, t)}{\phi(X, t)} = \frac{d}{dt} \sum_r \frac{(E_r)_{a,a}}{t - \theta_r} = -\sum_r \frac{(E_r)_{a,a}}{(t-\theta_r)^2} < 0 \quad □$$

Try graphing $\frac{t^2-1}{t^3-2t}$.



CO 444

23 Feb 2015

# Interlacing Again

If B is a principle submatrix of A, then $\theta_r(B) \le \theta_r(A)$.

We proved $\frac{d}{dt}\frac{\phi(X\backslash a,t)}{\phi(X,t)} < 0$, except of course at poles. (Fig. 1)

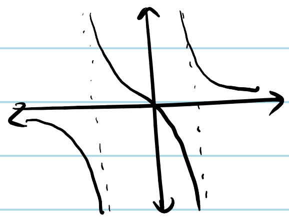


Fig. 1

**Lemma.** The Petersen graph does not have a Hamitonian cycle.

**Conjecture** (Lovácz). Every connected vertex-transitive graph has a Hamiltonian path.

But there is no known connected Cayley graph on more than two vertices that lacks a Hamiltonian cycle. The only vertex-transitive graphs without Hamiltonian cycles we know are $K_2$, Petersen, Coxeter, $L(S(\text{Petersen}))$, $L(S(\text{Coxeter}))$.

S(X) is X with each edge subdivided.

**Proof** of Lem (Mohar). Suppose Petersen has a Hamiltonian cycle C. Then $L(C) \cong C$ is an induced subgraph of L(Petersen). But by checking eigenvalues, $\theta_7(C) > \theta_7(L(\text{Petersen}))$, which fails interlacing. ■

Spectrum of P : $3^{(1)}\ 1^{(5)}\ -2^{(4)}$

$L(P)$ : $4^{(1)}\ 2^{(5)}\ -1^{(4)}\ -2^{(5)} \rightarrow \theta_7(L(P)) = -1$

$C_{10}$ : $2^{(1)}\ \frac{\pm1\pm\sqrt{5}}{2}^{(2)}\ -2^{(1)} \rightarrow \theta_7(C_{10}) = \frac{1-\sqrt{5}}{2} > -1$.

## Equitable Partitions

A partition $\pi$ of $V(X)$ with cells $C_1, \dots, C_m$ is equitable if each cell induces a regular subgraph and the edges joining two distinct cells make a semiregular bipartite graph. More precisely, for each i and j, the number of neighbours in $C_j$ of a vertex in $C_i$ is determined by i and j alone.

For example, an eq. part. on Petersen (Figure 2).

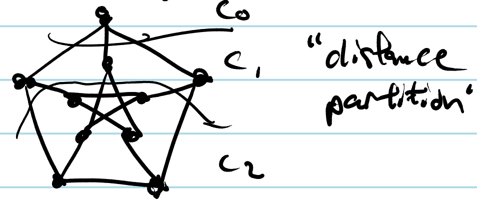
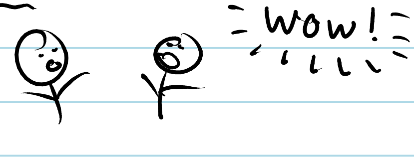


Fig. 2

If $\pi = (C_1, \dots, C_m)$ is a partition of X then the characteristic matrix of $\pi$ is the $V(X) \times m$ matrix that has the characteristic vectors of $C_1, \dots, C_m$ as its columns.

These columns are orthogonal and sum to $\underline{1}$.

To relate this to linear algebra, if X has adj. mat. A and eq. part. $\pi$ with char. mat. $\Pi$ then the product $A\Pi$ is constant on the cells of $\pi$. That is, $\text{Col}(\Pi)$ is A-invariant, or there exists B such that $A\Pi = \Pi B$. B is an $m \times m$ matrix, and $m \le V(X)$, so B has less eigenvalues What else is nice is that the eigenvalues of B are also eigenvalues of A.



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# Graph Isomorphism

Vectorize the adjacency matrix and check permutations of the vertices. Take the lexicographically smallest such vectorization. Tada! Isomorphism algorithm.

But some people like a more efficient treatment. Partition the graph by valency. This will almost never give a full algorithm, as at least one cell of this partition will have size > 1.

Now construct a profile for each vertex: how many vertices in each partition it is adjacent to, along with which partition the vx came from. Automorphisms must preserve profiles. Now partition by equivalence classes of profiles. Now go again! This process settles on the coarsest equitable partition refining the partition you started with.

## Normalized Characteristic Matrix.

Consider a partition $\pi$, char. mat. $P$. Define the normalized characteristic matrix $\hat{P}$ to be $P$ but with all columns scaled to have norm 1. Notice $P^TP=D$ where $D$ is a diagonal matrix (sizes of cells of $\pi$), so $\hat{P}=PD^{-\frac{1}{2}}$.

**Theorem.** The following are equivalent.

- (a) partition $\pi$ is equitable
- (b) $\operatorname{Col}(P)$ is $A$-invariant
- (c) $AP=PB$ for $B=A(X/\pi) \leftarrow$ not really an adj. mat. "quotient graph"
- (d) $\hat{P}\hat{P}^T$ commutes with $A$.

**Theorem.** If $\pi$ is equitable then $\phi(X/\pi,t)$ divides $\phi(X,t)$.

messy things start happening

Consider $AP=PB$. Take $[P\ Q]_{n\times n}$ s.t. $P^TQ=0$

$$A[P\ Q]=[P\ Q]\begin{bmatrix}B&0\\0&C\end{bmatrix}\Rightarrow\begin{bmatrix}P^TAP&0\\0&Q^TAQ\end{bmatrix}=\begin{bmatrix}P^T\\Q^T\end{bmatrix}A[P\ Q]=\begin{bmatrix}P^TP&0\\0&Q^TQ\end{bmatrix}\begin{bmatrix}B&\\&C\end{bmatrix}$$

Also notice $AP=PB\Rightarrow A^kP=PB^k\Rightarrow p(A)P=Pp(B)\Rightarrow$ min poly divides, and with somemore work you get char. poly.

These are some methods to prove Thm.

Also, if $Bz=\lambda z$, then $APz=PBz=\lambda Pz$ so $Pz$ is a $\lambda$-evec for $A$. Now suppose $Az=\lambda z$. Then $z^TA=\lambda z^T$ so $z^TPB=z^TAP=\lambda z^TP$ so $z^TP$ is a left evec for $B$.

Let's look at $A^kP=PB^k$ again. $P^TA^kP=P^TPB^k$ where $P^TP$ is diagonal. Suppose $a\in V(X)$ and $\{a\}$ is a cell of $\pi$. Then

$$(A^k)_{a,a}=(P^TA^kP)_{a,a}=(P^TPB^k)_{\{a\},\{a\}}=(B^k)_{\{a\},\{a\}}.$$

But also $\dfrac{\phi(X\backslash a,t)}{\phi(X,t)}=\dfrac{\phi(X/\pi\backslash a,t)}{\phi(X/\pi,t)}$ and $\sum_{a\in V(X)}\phi(X\backslash a,t)=\phi'(X,t)$. So if $X$ is walkregular ($\phi(X\backslash a,t)$ doesn't depend on $a$) then

$$|V(X)|\frac{\phi(X/\pi\backslash a,t)}{\phi(X/\pi,t)}=\frac{\phi'(X,t)}{\phi(X,t)}=\sum_{\theta\text{ eval}}\frac{\operatorname{mult}(\theta)}{t-\theta}. \longleftarrow \text{Q32?}$$

$$\frac{\phi(X\backslash a,t)}{\phi(X,t)}=(tI-A)^{-1}_{a,a}=\sum_r\frac{(E_r)_{a,a}}{t-\theta_r}\text{ so}$$

$$\sum_{a\in V(X)}\frac{\phi(X\backslash a,t)}{\phi(X,t)}=\sum_r\frac{\operatorname{tr}(E_r)}{t-\theta_r}=\sum_r\frac{\operatorname{mult}(\theta_r)}{t-\theta_r}=\frac{\phi'(X,t)}{\phi(X,t)}.$$

# Some Coding Theory

CO 444

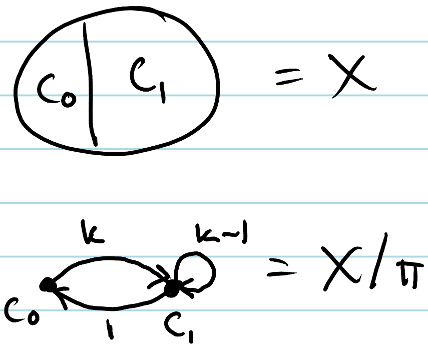
26 Feb 2015

To talk about coding theory, we need terminology.

The <u>ball of radius r</u> $B_r(x)$ about a vertex $x$ is the set of vertices at distance less than or equal to $r$ from $x$. For a <u>code</u> $C \subseteq V(X)$, the <u>packing radius</u> is the maximum integer $e$ such that the balls of radius $e$ about each vertex are disjoint, and the <u>covering radius</u> is the least integer $f$ such that the union of balls of radius $f$ about the vertices in $C$ covers $V(X)$. A code is <u>perfect</u> if its covering radius and packing radius are equal.

<u>Claim</u>. If $C$ is a perfect code in a regular graph, its distance partition is equitable.

For instance, suppose $C$ is a perfect 1-code on $X$, and $X$ is $k$-regular. Forming the quotient graph $X/(C_0, C_1)$, it is easy to see that the adjacency matrix of that quotient is $\begin{bmatrix} 0 & k \\ 1 & k-1 \end{bmatrix}$. (See Fig. 1) So the eigenvalues are $k$ and $-1$, and thus, if a graph has a perfect 1-code, it has an e/val $-1$.



<u>Fig. 1</u>

Some graphs which are interesting to coding theory are the Johnson graph $J(v,k)$ and the Kneser graph $Kn(v,k)$.

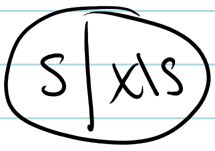
<u>Conjecture</u> (Desargue's Temptation). There is no perfect code in $J(v,k)$ with $v \geq 2k$ with more than two vertices.

There are perfect 1-codes on $K(7,3)$ and $K(11,5)$. No more are known.

## <u>Interlacing III</u>: Return of the Jedi.

$X$ a graph, $A = A(X)$, $\pi$ a partition of $V(X)$. Then $\pi$ partitions the rows and cols of $A$, so we define the <u>matrix of averages</u> to have entries the average of the row sums of the blocks of $A$. For instance, take a coclique $S$ of a $k$-regular $X$. (Fig. 2) Then,

$X =$



the matrix of averages is $\begin{bmatrix} 0 & k \\ \frac{k|S|}{|X\backslash S|} & k\left(1 - \frac{|S|}{|X\backslash S|}\right) \end{bmatrix}$.

$\pi = (S, X \backslash S)$

<u>Fig. 2</u>

<u>Claim</u>. The matrix of averages is $D^{-1}P^TAP$, where $D = P^TP$. <u>Pf</u>. Exer.

Note that $D^{-1}P^TAP = D^{-\frac{1}{2}}D^{-\frac{1}{2}}P^TAPD^{-\frac{1}{2}}D^{\frac{1}{2}} = D^{-\frac{1}{2}}Q^TAQD^{\frac{1}{2}}$.

Let $L = [Q \; R]_{n \times n}$ be orthogonal. Then $L^TAL = \begin{bmatrix} Q^T \\ R^T \end{bmatrix} A \begin{bmatrix} Q & R \end{bmatrix} = \begin{bmatrix} Q^TAQ & Q^TAR \\ R^TAQ & R^TAR \end{bmatrix}$. So $Q^TAQ$ is a principal submatrix of the symmetric matrix $L^TAL$, and we have interlacing.

Consider the coclique example again. If $\tau$ is the least eigenvalue of $A$ (nonzero if $A$ knows what's good for it) we have a bound $-\tau \geq \frac{k|S|}{|X\backslash S|}$, so taking $s = |S|$ and $v = |V(X)|$, we see:

$$(v-s)(-\tau) \geq ks \Rightarrow v(-\tau) \geq s(k-\tau) \Rightarrow s \leq \frac{v(-\tau)}{k-\tau} = \frac{v}{1-\frac{k}{\tau}}.$$

Applying this to Petersen we see $S \leq \frac{10}{1-\frac{3}{-2}} = 4$ and the bound is tight. Is it always tight?

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2 Mar 2015

# Eigenbounds

If $X$ is $k$-regular, with least eigenvalue $\tau$, then

$$\alpha(X) \le \frac{n}{1 - \frac{k}{\tau}} \quad \text{(The ratio bound.)}$$

## The Inertia Bound

Let $M$ be a real symmetric matrix. The inertia of $M$ is the triple $(n(+), n(0), n(-))$ of the numbers of positive, zero, and negative e/vals.

**Theorem.** (Cvetković). $\alpha(X) \le \min\{n - n(+), n - n(-)\}$. (The inertia bound)

For example, on Petersen, $n(+) = 6$ and $n(-) = 4$, so $\alpha(X) \le 4$. Also, on odd cycles, we get $\alpha(X) \le \lfloor \frac{n-1}{2} \rfloor$.

**Proof.** Let $C$ be a coclique with size $c$. Then by interlacing, $0 = \theta_r(C) \le \theta_r(A(X))$ for $1 \le r \le c$, so $\theta_c(A(X)) \ge 0$. Also, consider $\theta_c(-A(X))$ to see at least $c$ neg. e/vals. ☐

## Graph Products

We have the Cartesian product $X \square Y$ and the direct product $X \times Y$. This are related to the Kronecker product of matrices $A \otimes B$. Observe that

$$A(X \times Y) = A(X) \otimes A(Y) \quad \text{and} \quad A(X \square Y) = I \otimes A(X) + A(Y) \otimes I.$$

We look at some key properties of the Kronecker product.

$\rightarrow$ $(A \otimes B)(C \otimes D) = AC \otimes BD$, proviso $AC$ and $BD$ are defined. ①

$\rightarrow$ $\operatorname{vec}(AMB^T) = (B \otimes A)\operatorname{vec}(M)$ ②

Notice from ① that if $Ax = \lambda x$ and $By = \mu y$ then

$$(A \otimes B)(x \otimes y) = Ax \otimes By = \lambda x \otimes \mu y = \lambda\mu(x \otimes y).$$

So the eigenvalues of $A \otimes B$ are all possible products between eigenvalues of $A$ and of $B$. This holds for $A(X \times Y) = A(X) \otimes A(Y)$ as well. With a little more work, the eigenvalues of $X \square Y$ are all possible sums between e/vals of $X$ and $Y$. So e.g. this gives the e/vals of the $d$-cube $Q_d = K_2^{\square d}$ easily.

$$K_2 : -1^{(1)} \; 1^{(1)} \Rightarrow K_2 \square K_2 : -2^{(1)} \; 0^{(2)} \; 2^{(1)} \Rightarrow K_2^{\square 3} : -3^{(1)} -1^{(3)} \; 1^{(3)} \; 3^{(1)} \Rightarrow \text{etc.}$$

Recall that the Kneser graph $K_{v:k}$ is the graph on $k$-subsets of a $v$-set, with adjacency between disjoint subsets. If $v = 2k+1$, this is the odd-graph. The eigenvalues of $K_{2r+1:r}$ are $r+1, -r, r-2, -r+1, \ldots, (-1)^i(r+1-i)$ and $\binom{2r+1}{i} - \binom{2r+1}{i-1}$ (?)

Question. What is the maximum number of vertices in a bipartite subgraph of $K_{9:4}$?

Let $\alpha_2(X)$ denote the max number of vertices in a bipartite subgraph of $X$. Then,
**Claim.** $\alpha_2(X) = \alpha(X \square K_2)$. (Exer.)

| graph | spectrum | inertia |
|---|---|---|
| $K_{5:2}$ | $3^{(1)} \; -2^{(4)} \; 1^{(5)}$ | |
| $K_{7:3}$ | $4^{(1)} \; -3^{(6)} \; 2^{(14)} \; -1^{(14)}$ | |
| $K_{9:4}$ | $5^{(1)} \; -4^{(8)} \; 3^{(27)} \; -2^{(48)} \; 1^{(126)}$ | |
| $K_{5:2} \square K_2$ | $4^{(1)} \; 2^{(1)} \; -1^{(4)} \; -3^{(4)} \; 2^{(5)} \; 0^{(5)}$ | $(7, 5, 8)$ |

# Distance Regular Graphs

CO 444

4 Mar 2015

A graph $X$ is distance-regular if, for each triple $(i,j,r)$ and for each pair of vertices $a,b$ at distance $r$, the number of vertices at distance $i$ from $a$ and $j$ from $b$ is determined by $i$, $j$, and $r$. The number of such vertices is denoted $p_{i,j}(r)$ and is called the intersection number. (Consider the intersection of spheres of distances $i$ and $j$ around $a$ and $b$ respectively.)

A graph $X$ is distance-transitive if, for each $r$, $\mathrm{Aut}(X)$ is transitive on ordered pairs of vertices at distance $r$.

Examples: $K_n$, $K_{n,n}$, $C_n$, $J(v,k)$ for $v \geq 2k+1$, $H(n,q)$ ←

**Recall:**
Hamming graph
$V(H(n,q)) = Q^n$ for $|Q| = q$
and $u \sim v$ if at Ham. dist 1

If $X$ is a graph, the $i$-th distance graph $X_i$ is a graph with vertex set $V(X)$ and two vertices are adjacent if they are of distance $i$ in $X$. $X_1 = X$. Set $A_i = A(X_i)$ and $A_0 = I$.

We have that $\sum_{i=0}^{\mathrm{diam}(X)} A_i = J$. Something something commuting w/ matrix algebra.

If $X$ is distance-regular, $(A_i A_j)_{a,b} = \sum_r p_{i,j}(r) A_r$. One consequence is that $A_i$ commutes with $A_j$ for all $i$ and $j$.

Notice that $A_1 A_i = b_i A_{i-1} + a_i A_i + c_i A_{i+1}$. ← ?

E.g. $A_1^2 = kA_0 + a_1 A_1 + c_2 A_2$. It follows that $A_r = p_r(A_1)$ for some degree $r$ poly.

A distance-regular graph of diameter two is a strongly-regular graph. The converse is not generally true but w/e Godsil don't g.a.f.

Define the Paley graph for a finite field of $q$ elements, $q \equiv 1 \pmod 4$. Take $V(X) = \mathbb{F}_q$ and say $x \sim y$ iff $y-x$ is a square in $\mathbb{F} \backslash 0$. For instance, when $q=5$ we have $X = C_5$ the 5-cycle.

Define a symmetric design: a $v \times v$ 01-matrix $B$ such that

$$B\mathbf{1} = k\mathbf{1},\quad \mathbf{1}^T B = k\mathbf{1}^T, \text{ and } BB^T = nI + \lambda J \quad \text{for } k, n, \lambda.$$

E.g. $A = \begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix}$ and it has diameter three or something?

# More Distance-Regularity

CO 444

5 Mar 2015

Consider the distance partition on a distance-regular graph. (See Figure 1) This partition is equitable because the number of vertices at distance $i+1$ from $u$ and 1 from a distance-$i$ vertex $v$ is determined by the graph, and likewise for neighbours in distance cells $i$ and $i+1$. Denote the distance partition relative to $u$ by $\pi_u$.

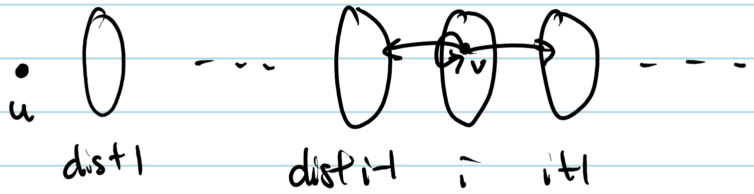

Fig.1 Dist. partition rel. to u.

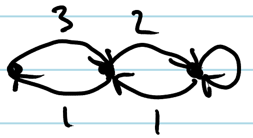

Fig. 2 Petersen/$\pi_u$

Then $X/\pi_u$ is a nonsimple path. (E.g. Figure 2) The eigenvalues of $X/\pi_u$ are all e/vals of $X$, and also are all simple. If $B$ is $A(X/\pi_u)$ then $(A^k)_{u,u} = (B^k)_{\{u\},\{u\}}$ for all $u$, so then

$$\frac{\phi(X\backslash u,t)}{\phi(X,t)} = \frac{\phi(X/\pi_u\backslash\{u\},t)}{\phi(X/\pi_u,t)} \Rightarrow \frac{\phi'(X,t)}{\phi(X,t)} = \sum_{v\in X}\frac{\phi(X\backslash v,t)}{\phi(X,t)} = |V(X)|\frac{\phi(X/\pi_u\backslash\{u\},t)}{\phi(X/\pi_u,t)},$$

where every $X/\pi_u$ is isomorphic to every other by distance-regularity.

## Symmetric Designs

Recall: a $v\times v$ 01-matrix $B$ such that $B\mathbf{1} = k\mathbf{1} = B^T\mathbf{1}$ and $BB^T = (k-\lambda)I - \lambda J$.

$$A^2 = \begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix}^2 = \begin{bmatrix} B^TB & 0 \\ 0 & BB^T \end{bmatrix} = \begin{bmatrix} (k-\lambda)I+\lambda J & 0 \\ 0 & (k-\lambda)I+\lambda J \end{bmatrix} = (k-\lambda+1)I + \underbrace{\begin{bmatrix} J-I & 0 \\ 0 & J-I \end{bmatrix}}_{A_2}$$

$$A_1A_2 = \begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix}\begin{bmatrix} J-I & 0 \\ 0 & J-I \end{bmatrix} = \begin{bmatrix} 0 & kJ-B^T \\ kJ-B & 0 \end{bmatrix} = ?$$

Set $A_3 = \begin{bmatrix} 0 & J-B^T \\ J-B & 0 \end{bmatrix}$. Then from the above, $A_3$ is a poly in $A_1$ of degree three. We conclude that $X$ is distance-regular with diameter 3.

We can see that $A$ has exactly four eigenvalues, say $\theta_0 \geq \theta_1 \geq \theta_2 \geq \theta_3$. Since $X$ has valency $k$, $\theta_0 = k$, and it is simple since $X$ is connected. $X$ is bipartite so the spectrum is symmetric about the origin, so $\theta_3 = -k$ is simple, and $\theta_2 = -\theta_1$ with each having multiplicity $v-1$.

We know $\sum_{\theta \text{ e/val}} m_\theta = v$, $\sum_{\theta \text{ e/val}} \theta m_\theta = 0$, and $\sum_{\theta \text{ e/val}} \theta^2 m_\theta = vk$, so we can find that

$$k^2 + (v-1)\theta_1^2 = vk \Rightarrow \theta_1^2 = \frac{vk-k^2}{v-1} = k-\lambda \Rightarrow \theta_1 = \sqrt{k-\lambda}.$$

### Walk-regular Graphs

A graph $X$ is walk-regular if the generating function for closed walks at a vertex is independent of the vertex. E.g. vertex-transitive and distance-regular graphs are walk-regular.

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# Laplacian Matrix

The Laplacian matrix of a graph $X$ is $L = \Delta - A = \hat{B}\hat{B}^T$ where $\hat{B}$ is the incidence matrix of an orientation of $X$.

## Properties of the Laplacian

- → Eigenvalues of $L$ are nonnegative. Letting $\lambda$ an e/val and $x$ its e/vec,

$$\lambda \|x\|^2 = \lambda x^T x = x^T L x = x^T N N^T x = (N^T x)^T (N^T x) = \|N^T x\|^2 \geq 0$$

- → $0$ is an e/val of $L$ and its multiplicity is the number of connected components. $\mathbb{1}$ is an eigenvector for $0$.

- → For any vector $x$, $x^T L x = \sum_{u \sim v} (x_u - x_v)^2 \geq 0$. Follows from $x^T L x = \|N^T x\|^2$.

- → Any e/val is at most $|V(X)|$. $L(X) + L(\bar{X}) = nI - J$.

- → $L$ is positive semidefinite and not positive definite. Recall that for $M$ a symmetric matrix, the following are equivalent:
  - (a) $M$ has nonneg e/vals
  - (b) $M = BB^T$
  - (c) $x^T M x \geq 0$
  - (d) eigenvalues of any principal submatrix are nonnegative
  - (e) determinant of any principal submatrix are nonneg.
  - (f) $\det(tI - M) = t^{\text{null}(M)} p(t)$ where the coefficients of $p$ have alternating signs.

## Counting Spanning Trees.

Given a graph $X$ and $e \in E(X)$, define $X \backslash e$ ($X$ remove $e$) and $X/e$ ($X$ contract $e$). If $\tau(X)$ is the number of spanning trees of $X$, then $\tau(X) = \tau(X \backslash e) + \tau(X/e)$. (Pf. $\tau(X \backslash e)$ counts the number of trees not containing $e$, $\tau(X/e)$ counts ones that do.)

**Theorem** (Kirchoff). Let $X$ be a graph and $u \in V(X)$. Then $\tau(X) = \det L(u|u)$.

**Proof.** Proceed by induction on $|E(X)|$. Let $e = \{u,v\} \in E(X)$. Let $E = e_v e_v^T$. Then $L(X) = L(X \backslash e)(u|u) + E$. (Argue by considering the matrix.) Thus, $\det L(X)(u|u) = \det L(X \backslash e)(u|u) + \det(X \backslash e)(u,v|u,v)$. But notice that $L(X \backslash e)(u,v|u,v) = L(X)(u,v|u,v) = L(X/e)(u|u)$. So then by induction,

$$\det L(X)(u|u) = \det L(X \backslash e)(u|u) + \det(X/e)(u|u) = \tau(X \backslash e) + \tau(X/e) = \tau(X). \blacksquare$$

**Corollary.** $\tau(X) = \frac{1}{n} \prod_{i \geq 2} \lambda_i$ where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are e/vals of $L(X)$. ∎

This is sometimes called the "Matrix-Tree Theorem".

# Quantum Walks

CO 444

11 Mar 2015

Whereas a classical bit is a selection out of $\{0,1\}$, a qubit is a 1-dimensional subspace of $\mathbb{C}^2$. We consider a system consisting of a graph with a qubit assigned to each vx.

The state of this system is determined by $U(t) = \exp(itL)$ for $L$ the Laplacian, $i$ the imaginary unit, and exp the matrix exponential. So

$$U(t) = \exp(itL) = \sum_{\lambda \text{ e/val}} \exp(it\lambda) E_\lambda = \sum_k \frac{1}{k!}(it)^k L^k.$$

For example, taking $X = K_2$, $L = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ so $L^k = 2^{k-1}L$, and then

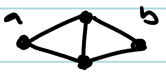
$$U(t) = I + \frac{1}{2}(e^{2it} - 1)L.$$

Observe that:

$$\to U(t)^T = U(t) \qquad \to U(t)^* = U(t)^{-1} \qquad \to U(t)U(s) = U(t+s)$$

→ $U(t)$ is unitary, so its rows determine a probability distribution. In particular $|U(t)_{a,b}|^2$ is the probability that a state input of $a$ is seen at $b$ after time $t$.

The interesting question is perfect state transfer: when is $|U(t)_{a,b}|^2 = 1$? Equivalently, when is $U(t)e_a = \gamma e_b$? Say $t$ is the time, and $\gamma$ is the phase

For example, take $X =$ . Then

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \text{ so } U(t) = e^{4it}\frac{1}{4}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} + e^{2it}\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + e^{0it}\frac{1}{4}J,$$

and $U(\frac{\pi}{2})e_a = e_b$. In general, $U(t)e_a = \gamma e_b$ iff $e^{it\lambda}E_\lambda e_a = \gamma E_\lambda e_b$ for all e/vals $\lambda$.

So we can define ab-pst on a graph to be that there exists a $t > 0$ such that $e^{it\lambda}E_\lambda e_a = \gamma E_\lambda e_b$ for all $\lambda$.

Notice that if we use the Laplacian, 0 is always an eigenvalue with $E_0 = \frac{1}{n}J$, so since $e^{0it} = 1$, $E_0 e_a = E_0 e_b$ and thus $\gamma = 1$. For all $\lambda$, $E_\lambda e_a$ and $E_\lambda e_b$ must be real valued. If $E_\lambda e_a \neq 0$, then $e^{\lambda it} = \pm 1$.

Hence a necessary condition for pst is that $E_\lambda e_a = \pm E_\lambda e_b$ for all e/vals $\lambda$. Vertices a and b satisfying this are Laplacian-strongly-cospectral.

Let $\Lambda^+_{ab}$ denote the set of e/vals $\lambda$ s.t. $E_\lambda e_a = E_\lambda e_b$ and let $\Lambda^-_{ab}$ be those that have $E_\lambda e_a = -E_\lambda e_b$. Pst occurs at time $\tau$ if

$$e^{i\tau\lambda} = 1 \text{ if } \lambda \in \Lambda^+ \text{ and } e^{i\tau\lambda} = -1 \text{ if } \lambda \in \Lambda^-.$$

Equivalently $\lambda \in \Lambda^+$ (resp $\Lambda^-$) means $\lambda t$ is an even (resp. odd) multiple of $\pi$.

# More on Quantum Walks

CO 444

12 Mar 2015

let $X$ be a graph, $a,b \in V(X)$. When is $e_a - e_b$ an eigenvector for the Laplacian? If $Lx = \lambda x$ then $(\deg c) x_c - \sum_{w \sim c} x_w = \lambda x_c$. In particular $\sum_{w \sim c} x_w = 0$ and $\lambda = \deg c$. But this means if $a \sim c$ then $b \sim c$ and vice-versa. Pairs of vertices like this are called twins.

**Theorem.** $X$ admits ab-pst iff (1) $E_\lambda e_a = \pm E_\lambda e_b$ for all e/vals $\lambda$ of $L$, (2) if $\lambda \in \Lambda^+_{ab} \cup \Lambda^-_{ab}$ — meaning $E_\lambda e_a \neq 0$ — then $\lambda \in \mathbb{Z}$ (this is hard!), and (3) there exists an $\alpha$ such that $\lambda / 2^\alpha$ if even if $\lambda \in \Lambda^+_{ab}$ and odd if $\lambda \in \Lambda^-_{ab}$ ▨

**Theorem.** (C., Liu?) There is no Laplacian-pst on trees with more than two vertices.

Recall that $\tau(X) = \det L(X)(u|u)$. Trees have $\tau(X) = 1$ so $\det L(u|u) = 1$ for any $u$. rank $L = n-1$ by linear algebra, and moreover rank $L = n-1$ over any field of prime order $\mathbb{Z}_p$.

**Theorem.** Let $X$ be a tree and $a,b \in V(X)$. If $\lambda$ is an integral Laplacian eigenvalue in $\Lambda^-_{ab}$ then $\lambda$ is a power of 2.

**Proof** Let $p$ be prime dividing $\lambda$. Let $v$ be a $\lambda$-e/vector. Then $v$ is integral in its entries, and we can choose the gcd of the entries to be 1. Mod $p$, $Lv = \lambda v \equiv 0$, so since $\dim \ker L = 1$, $v$ is a scalar multiple of $\mathbf{1}$ mod $p$. But $E_\lambda e_a = -E_\lambda e_b$, so $v^T e_a = v^T E_\lambda e_a = -v^T E_\lambda e_b = -v^T e_b$. So if $v \equiv k\mathbf{1}$ mod $p$, then $k \equiv -k$ mod $p$, so since $k \neq 0$, $p = 2$. Hence 2 is the only prime dividing $\lambda$. ▨

**Corollary.** If a tree admits ab-pst, there is at most one eigenvalue in $\Lambda^-_{ab}$.

**Proof** Suppose $\Lambda^-_{ab} = \{\lambda, \mu\}$. They are distinct powers of 2. But there exists an $\alpha$ such that $\lambda/2^\alpha$ and $\mu/2^\alpha$ are odd. Contradiction. ▨

**Theorem.** No Laplacian-pst on trees of more than 2 vertices.

**Proof** Say $\Lambda^-_{ab} = \{\lambda\}$, so $E_\lambda e_a = -E_\lambda e_b$. Then we have that

$$(1) \sum_{\mu \in \Lambda^+_{ab}} E_\mu e_a + E_\lambda e_a = e_a, \text{ and}$$

$$(2) \sum_{\mu \in \Lambda^+_{ab}} E_\mu e_a - E_\lambda e_a = \sum_{\mu \in \Lambda^+_{ab}} E_\mu e_b + E_\lambda e_b = e_b$$

So $\sum_\lambda E_\lambda e_a = \frac{1}{2}(e_a + e_b)$ and $E_\lambda e_a = \frac{1}{2}(e_a - e_b)$. Hence $a,b$ are twins, not neighbours, and share a unique neighbour $c$. Then

$$0 = \sum_{\lambda \in \Lambda^+} e_c^T E_\lambda e_a = \sum_{\lambda \in \Lambda^+} (1-\lambda) e_a^T E_\lambda e_a \leq \frac{1}{n} - \sum_{\substack{\lambda \in \Lambda^+ \\ \lambda > 0}} e_a^T E_\lambda e_a = \frac{2}{n} - \sum_{\lambda \in \Lambda^+} = \frac{2}{n} - \frac{1}{2}.$$

So $n$ can be at most 4. Then manually verify there is no pst if $n = 3$ or $4$. ▨

# Continous Quantum Walks

CO 444

16 Mar 2015

We will now consider quantum walks involving the usual adjacency matrix $A(X)$. Define the ~~transition~~ matrix $U(t) = \exp(itA)$.

We have perfect state transfer from a vertex $a$ to a vertex $b$ at time if for some scalar $\gamma$, $U(t)e_a = \gamma e_b$.

We have $U(t)^T = U(t)$, $\overline{U(t)} = U(-t) = U(t)^{-1}$, $U(t)^* = U(t)^{-1}$ so $U$ is unitary. Hence $|\gamma| = 1$. We no longer have the Laplacian advantage of $\gamma = 1$.

If $U(t)e_a = \gamma e_b$ then $U(t)_{b,a} = e_b^T U(t) e_a = \gamma$. Thus $|U(t)_{b,a}| = 1$, which by the symmetry property means $|U(t)_{a,b}| = 1$, and by the unitary property means every other entry in that row or column is $0$.

Take e.g. $X = K_2$, so $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $A^2 = I$. Then

$$U(t) = I + itA + \frac{(it)^2}{2}A^2 + \frac{(it)^3}{3!}A^3 + \dots = \cos t\, I + i\sin t\, A = \begin{bmatrix} \cos t & i\sin t \\ i\sin t & \cos t \end{bmatrix}$$

So $U(\frac{\pi}{2}) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = iA$. So we have pst from $0$ to $1$ at $\frac{\pi}{2}$ with phase factor $i$.

$U(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$, and we call such a matrix flat, if the magnitude of each entry is equal to 1.

We also have pst from 1 to 3 on $P_3$. To see this, look at the characteristic polynomial. $\phi(P_3, t) = t^3 - 2t$ so the spectrum is $0, \pm\sqrt{2}$. Then

$$A = \sqrt{2}E_{\sqrt{2}} + 0E_0 - \sqrt{2}E_{-\sqrt{2}} \quad \text{and} \quad U(t) = e^{it\sqrt{2}}E_{\sqrt{2}} + e^{-it\sqrt{2}}E_{-\sqrt{2}} + E_0$$

Recall that for simple e/vals, $E_\lambda = zz^T$ for $Az = \lambda z$ and $\|z\| = 1$. Also recall $E_{\sqrt{2}} + E_{-\sqrt{2}} + E_0 = I$ and $E_{\sqrt{2}} - E_{-\sqrt{2}} + E_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ The full proof is in Godsil's notes.

Pst fails on $P_4$. You can get "Pretty Good state transfer" on $P_4$, $P_5$, $P_6$, and $P_7$, but it's hard and fails when you reach $P_8$. We want more pst graphs, though.

Recall the Cartesian product $X \square Y$ is such that $A(X \square Y) = I \otimes A(X) + A(Y) \otimes I$. Notice that the summands commute. If $MN = NM$, $\exp(M+N) = \exp(M)\exp(N)$

$$\begin{aligned} U_{X\square Y}(t) &= \exp(it\, I \otimes A(X) + it\, A(Y) \otimes I) = \exp(it\, I \otimes A(X))\exp(it\, A(Y) \otimes I) \\ &= (I \otimes \exp(itA(X)))(\exp(itA(Y)) \otimes I) = U_Y(t) \otimes U_X(t). \end{aligned}$$

This is cool in and of itself. But also, notice the d-cube $Q_d = K_2^{\square d}$. So then

$$U_{Q_d}(t) = U_{K_2}(t)^{\otimes d} \quad \text{and} \quad U_{Q_d}(\tfrac{\pi}{2}) = i^d \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\otimes d}.$$

A similar infinite family can be derived from $P_3^{\square d}$. Are there any more?

If $X = K_n$ the complete graph, $A = J - I = (n-1)\frac{1}{n}J + (-1)(I - \frac{1}{n}J)$, so

$$U(t) = \frac{1}{n}e^{(n-1)it}J + e^{-it}(I - \tfrac{1}{n}J) = e^{-it}\left(\frac{1}{n}e^{int}J + I - \frac{1}{n}J\right) = e^{-it}\left(I - \frac{1-e^{int}}{n}J\right)$$

which acts a lot like the identity. "The quantum walkers tend to stay at home."

Recall that the distance partition on the d-cube is equitable. You can derive that $U(t)$ is constant on the cells of this partition, so you can work on the quotient graph of that partition, a weighted path. (Not usually done.)

The moral is that pst is hard.

CO 444

18 Mar 2015

# Periodicity

Suppose we have ab-pst at time $t$ with phase $\gamma$. Then $U(t)e_a = \gamma e_b$ and $U(t)e_b = \gamma e_a$, so

$$U(2t)e_a = U(t)^2 e_a = U(t)\gamma e_b = \gamma^2 e_a.$$

If there is a time $t$ and a scalar $\gamma$ so that $U(t)e_a = \gamma e_a$, we say $X$ is periodic at $a$. A graph is periodic if it is periodic at each vertex at time $t$.

If $X$ is periodic at time $t$ then $U(t)$ is diagonal. If $X$ is also connected then $U(t)$ is a scalar matrix $\gamma I$.

interesting fact: If $X$ has ab-pst at time $t$ and ac-pst at time $s$, then $s=t$ and hence $b=c$. This is pretty hard to prove and not usable on assignments, but something to keep in mind nonetheless.

Lemma. If $n = |V(X)|$ and $U(t) = \gamma I$ then $\gamma^n = 1$.

Proof. For any square matrix $M$, $\det \exp M = \exp \operatorname{tr} M$. Then

$$\gamma^n = \det \gamma I = \det U(t) = \det \exp(itA) = \exp \operatorname{tr}(itA) = \exp 0 = 1. \quad ▨$$

## Cubelike Graphs

If $X$ is a cubelike with valency $m$ then $A = P_1 + \cdots + P_m$ where $P_i$ is the adjacency matrix of a perfect matching. We have $P_r^2 = I$, $P_rP_s = P_sP_r$, and $\operatorname{tr} P_r = 0$. By $P^2 = I$ we have $U_P(t) = \exp(itP) = \cos t\, I + i \sin t\, P$. Then

$$U_A(t) = \prod_{r=1}^{m} U_P(t) = \prod_{r=1}^{m} (\cos t\, I + i \sin t\, P) \quad \text{and} \quad U_A(\tfrac{\pi}{2}) = i^m \prod_{r=1}^{m} P_r.$$

This product of permutation matrices is itself a permutation matrix, and if it is not the identity we have pst, and in fact the diagonal is zero. Hence we see that if $X = X(\mathbb{Z}_2^d, C)$ is cubelike and $\sum C \neq 0$ then there is pst on $X$ from $\underline{0}$ to $\sum C$.

## Integer Eigenvalues

If the eigenvalues of $A$ are integers, then $U(2\pi) = \sum_\lambda e^{2\pi i\lambda} E_\lambda = \sum_\lambda E_\lambda = I$, so $X$ is periodic. Eventually we will find that if vertex-transitive graphs have pst, they must have integer eigenvalues. (And vice-versa maybe?)

Theorem. If $X$ is periodic, its eigenvalues are integers.

Proof. Suppose $U(t)_{a,a} = \gamma$, and $|\gamma| = 1$. We have that

$$U(t)_{a,a} = \sum_\lambda e^{it\lambda}(E_\lambda)_{a,a} \quad \text{where} \quad \sum_\lambda (E_\lambda)_{a,a} = 1,$$

so we have a convex combination of roots of unity. But saying $|\gamma| = 1$, then it must be exactly one of those points. It follows that $e^{it\lambda} = \gamma$ for all $\lambda$, so $e^{it(\lambda-\mu)} = 1$ for $\lambda, \mu$ e/vals.

$$t(\theta - \lambda) = 2m_\lambda \pi \quad \text{so} \quad \frac{\theta-\lambda}{\theta-\mu} = \frac{m_\lambda}{m_\mu} \in \mathbb{Q}.$$

(?)

The eigenvalues of a graph are algebraic integers, so if they are rational they are integers. ▨

# Continuous Quantum Walks, Continued

CO 444

19 Mar 2015

If $X$ is vertex-transitive, and there is $ab$-pst at time $t$,

$$U(t)_{a,a}=0 \Rightarrow U(t)_{u,u}=0 \text{ for all } u\in V(X).$$

Suppose $P\in \mathrm{Aut}(X)$ and $Pe_a=e_c$ and $Pe_b=e_d$. Then

$$\gamma e_d=\gamma Pe_b=PU(t)e_a=U(t)Pe_a=U(t)e_c$$

so there is $cd$-pst at the same time and with the same phase factor. It follows, without loss of generality, that

$$U(t)=\gamma\begin{bmatrix} \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & & \\ & \ddots & \\ & & \begin{smallmatrix}0&1\\1&0\end{smallmatrix}\end{bmatrix}=\gamma T \leftarrow ② \text{ what is going on here?}$$

Corollary. If we have pst on a connected vertex-transitive graph $X$, $|V(X)|$ is even.

Lemma. If $a,b\in V(X)$ and $X$ has $ab$-pst, $\mathrm{Aut}(X)_a=\mathrm{Aut}(X)_b$.

Pf. Suppose $U(t)e_a=\gamma e_b$ and $P\in\mathrm{Aut}(X)_a$. Then

$$\gamma Pe_b=PU(t)e_a=U(t)Pe_a=U(t)e_a=\gamma e_b \Rightarrow e_b=Pe_b. \blacksquare$$

## Integrality

We saw that if $X$ is periodic at $a$, then $\frac{\theta_1-\theta_r}{\theta_1-\theta_s}\in\mathbb{Q}$ for all eigenvalues of $X$ that matter.

For example, $\phi(P_4,t)=t^4-3t^2+1=(t^2-t-1)(t^2+t-1)$ so the spectrum of $P_4$ is $\frac{1}{2}(\pm1\pm\sqrt{5})$. The ratio condition does not hold, so no periodicity and thus no pst.

Theorem. If a vertex-transitive graph $X$ admits pst, its eigenvalues are integral.

Proof. Invoke Galois theory? If the ratio condition holds, we have $\frac{\theta_1-\theta_r}{\theta_1-\theta_s}\in\mathbb{Q}$ for all $a,b,r,s\neq r$. So

$$\frac{\theta_r-\theta_s}{\theta_1-\theta_2}\in\mathbb{Q} \Rightarrow \mathbb{Q}\ni\prod_{r\neq s}\frac{\theta_r-\theta_s}{\theta_1-\theta_2}=\frac{\prod_{r\neq s}(\theta_r-\theta_s)}{(\theta_1-\theta_2)^{\delta^2-\delta}} \text{ for } \delta=|\{\theta_r \mid r\}|.$$

The numerator is an integer and is fixed by any permutation of the roots of $\phi(X,t)$. It follows that $(\theta_1-\theta_2)^{\delta^2-\delta}\in\mathbb{Q}$. Since $\theta_1-\theta_2$ is an algebraic integer, then so is $(\theta_1-\theta_2)^{\delta^2-\delta}$, meaning $\theta_1-\theta_2$ is the $m$-th root of an integer. Since it is real, $m\leq 2$. It follows $(\theta_r-\theta_s)^2\in\mathbb{Z}$ for all $r,s$. We consider two cases:

(i) $\theta_1-\theta_2\in\mathbb{Z}$. By the ratio condition, $\theta_r-\theta_s\in\mathbb{Z}$, so we can write $\theta_r=\theta_1+v_r$ where $v_r\in\mathbb{Z}$. Then $\mathbb{Z}\ni\sum_r\theta_r=\sum_r\theta_1+v_r=n\theta_1+\sum_r v_r$. So $\theta_1\in\mathbb{Q}$, meaning $\theta_1$ is an integer, and so is every other eigenvalue.

(ii) $\theta_1-\theta_2\notin\mathbb{Z}$. Assume $\theta_1$ is the spectral radius, and let $\Delta=(\theta_1-\theta_2)^2$. Then or not $\theta_2=\theta_1-\sqrt{\Delta}$. But then $\theta_1+\sqrt{\Delta}$ is also a root of $\phi(X,t)$, which is a contradiction? More generally $\theta_r=\theta_1-\mu_r\sqrt{\Delta}$ for $\mu_r\in\mathbb{Z}$. After some work we deduce $\theta_r=a_r+b_r\sqrt{\Delta}$ where $a_r,b_r\in\frac{1}{2}\mathbb{Z}$.

# First, Some Loose Ends

CO 444

23 Mar 2015

Recall that an algebraic integer is a zero of a monic polynomial with integer coefficients.

If $X$ is periodic at $u$, its eigenvalues can be written in the form $\frac{1}{2}(a+b_r\sqrt{\Delta})$ where $a, b, \Delta \in \mathbb{Z}$. Exercise. If $\Delta$ is not a perfect square and $\frac{1}{2}(a+b_r\sqrt{\Delta})$ is an algebraic integer then $\Delta \equiv 1 \bmod 4$. Cor. Any two distinct eigenvalues differ by at least 1. Remark. If $X$ is regular, e/vals are integers.

## Dual Degree and Covering Radius

Lemma. If $X$ has diameter $d$ and exactly $s$ different eigenvalues, then $d+1 \leq s$.

Proof. The matrices $I, I+A, (I+A)^2, \ldots, (I+A)^d$ are linearly independent, so $\mathbb{R}[A]$ — the space of real polynomials in $A$ — has dimension at least $d+1$. But the spectral idempotents of $A$ form a basis for $\mathbb{R}[A]$ of size $s$. ▨

We introduce two interesting parameters. Suppose $S \subseteq V(X)$. The covering radius of $S$ is the maximum distance of a vertex in $X$ from $S$. If $S = \{u\}$, this is known as the eccentricity of $u$.

Let $\chi$ be the characteristic vector of $S$ in $X$. If $S$ has covering radius $r$, then the vectors $\chi, (I+A)\chi, \ldots, (I+A)^r\chi$ are linearly independent. Then if $E_1, \ldots, E_s$ are the spectral idempotents, then $E_1\chi, \ldots, E_s\chi$ are orthogonal, so the nonzero projections of $\chi$ are linearly independent. Since every $A^k\chi$ is a linear combination of $E_i\chi$, it follows that $r+1 \leq s$. This generalizes the Lemma above.

Define the eigenvalue support of $S$ to be $\{\lambda \mid \lambda \text{ e/val of } X, E_\lambda\chi_S \neq 0\}$. If $X$ is connected, the spectral radius is in the support. The dual degree of $S$ is one less than the size of the support. So we have that covering radius $\leq$ dual degree, and we will apply this to quantum walks. The key is to notice that $U(t)e_a$ lies in span$\{A^k e_a\}$. (In fact span$\{U(t)e_a \mid t \geq 0\}$ = span$\{A^k e_a\}$.)

We have $U(t)_{a,a} = \sum_\lambda e^{it\lambda}(E_\lambda)_{a,a}$. The number of nonzero summands is the size of the eigenvalue support of $\{a\}$. Recall that if $X$ is vertex-transitive (or just walk-reg.) then the eigenvalue support of a vertex is the set of all eigenvalues of $X$.

Theorem. There are only finitely many connected graphs with maximum valency at most $k$ on which pst occurs.

Proof. Let $\mathcal{U}$ be the set of connected graphs with pst and maximum valency at most $k$. Suppose pst occurs at $a$. Any two distinct eigenvalues in the eigenvalue support of $a$ differ by at least 1. All eigenvalues of $X$ lie in $[-k, k]$, so the dual degree of $a$ is at most $2k$, bounding the covering radius — i.e. the eccentricity — similarly. Therefore the diameter, and hence $|V(X)|$, is bounded by a function of $k$. ▨

We see from the above that there is a bound on the length of a path with pst. There also some fun things to be done with e.g. assuming there is pst on the central pair and working outwards.

CO 444

25 Mar 2015

# Uniform Mixing

We say there is uniform mixing at time $t$ if $U(t)$ is flat. A Hadamard matrix is an $n\times n$ $\pm1$-matrix $H$ such that $H^TH=nI$. So it is flat and $\frac{1}{\sqrt{n}}H$ is orthogonal. If $U(t)$ was real and had uniform mixing, it would be a scalar multiple of a Hadamard matrix. For example $U_{K_2}(\frac{\pi}{4})=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ i & 1\end{bmatrix}$ which is flat, so $K_2^{\square d}=Q_d$ has uniform mixing. There is also uniform $K_3$ and $K_4$, but no higher complete graphs.

## Bipartite graphs

If $X$ is bipartite, then $A=\begin{bmatrix}0 & B\\ B^T & 0\end{bmatrix}$, so $A^2=\begin{bmatrix}BB^T & 0\\ 0 & B^TB\end{bmatrix}$, $A^3=\begin{bmatrix}0 & BB^TB\\ B^TBB^T & 0\end{bmatrix}$, ... So

$$\exp(itA)=I+it\begin{bmatrix}0 & B\\ B^T & 0\end{bmatrix}+\frac{(it)^2}{2}\begin{bmatrix}BB^T & 0\\ 0 & B^TB\end{bmatrix}+\frac{(it)^3}{6}\begin{bmatrix}0 & BB^TB\\ B^TBB^T & 0\end{bmatrix}e\dots$$

$$=\left(I-t^2\begin{bmatrix}BB^T & 0\\ 0 & B^TB\end{bmatrix}+t^4\begin{bmatrix}(BB^T)^2 & 0\\ 0 & (B^TB)^2\end{bmatrix}-\dots\right)+i\left(t\begin{bmatrix}0 & BB^TB\\ B^TBB^T & 0\end{bmatrix}-\dots\right)=\begin{bmatrix}C_1(t) & iK(t)\\ iK(t)^T & C_2(t)\end{bmatrix},$$

where $C_1,C_2$ are symmetric and $C_1,C_2,K$ are real. Then let $H$ be the matrix

$$H=\begin{bmatrix}iI & 0\\ 0 & I\end{bmatrix}U(t)\begin{bmatrix}-iI & 0\\ 0 & I\end{bmatrix}=\begin{bmatrix}C_1 & -K\\ K^T & C_2\end{bmatrix},$$

which is real and orthogonal. If it is also flat, then $\frac{1}{\sqrt{n}}H$ is Hadamard.

**Lemma.** If an $n\times n$ Hadamard matrix exists, then $n=2$ or $4$ divides $n$. ▨

**Corollary.** $C_{4m+2}$ for $m\geq1$ does not admit uniform mixing. ▨

In fact, no even cycles admit uniform mixing and neither do cycles of prime order. There is some crazy number-theoretic stuff going on.

## Expanders and Ramanujan Graphs

We discuss random walks on a graph.

Let $L$ be the Laplacian of $X$. A continuous random walk is specified by

$$Q(t)=\exp(-tL)=I-t(\Delta-A)+\frac{t^2}{2}(\Delta-A)^2-\dots$$

$Q(t)\mathbf{1}=\mathbf{1}$ and $Q(t)\geq0$. The intuition is that we move to a neighbour with probability proportional to time spent on a vertex.

Let $P=\Delta^{-1}A$, assuming no vertex is isolated. A discrete random walk is given by $P^n$, and $(P^n)_{a,b}$ is the probability that, starting at $a$, we travel to $b$ after $n$ steps.

We consider the long term behaviour of these walks.

- In the continuous case, recall $L$ has the spectral decomposition $\sum_{r=1}^{m}\lambda_rF_r$. If $L$ is connected, $\lambda_1=0$ is a simple e/val and $F_1=\frac{1}{n}J$. So
$$Q(t)=\sum_{r=1}^{m}e^{-\lambda_rt}F_r=F_1+\sum_{r=2}^{m}e^{-\lambda_rt}F_r=\frac{1}{n}J+e^{-\lambda_2t}\sum_{r=2}^{m}e^{(\lambda_2-\lambda_r)t}F_r\to\frac{1}{n}J$$
in the limit, with $\lambda_2$ controlling how quickly it converges.
- In the discrete case, notice $\Delta^{-1}A=\Delta^{-\frac{1}{2}}(\Delta^{-\frac{1}{2}}A\Delta^{-\frac{1}{2}})\Delta^{\frac{1}{2}}$ is similar to $\Delta^{-\frac{1}{2}}A\Delta^{-\frac{1}{2}}$, the normalized Laplacian, which is symmetric. Call this $\hat{A}=\sum_r\theta_rE_r$. Consider momentarily the regular case. $\mathbf{1}$ is an e/vec for $\Delta^{-1}A$ so its eigenvalue is in the spectrum of $\hat{A}$.

# Isoperimetric Constant (Conductance)

CO 444

26 Mar 2015

If $S \subseteq V(X)$ then $\partial S$ is the set of edges with exactly one endpoint in $S$. The isoperimetric constant $h(X)$ is defined by

$$h(X) = \min\left\{ \frac{|\partial S|}{|S|} \;\middle|\; S \subseteq V(X),\ S \neq \emptyset,\ |S| \leq |V \setminus S| \right\}.$$

Remark. $\partial S = \partial(V \setminus S)$.

Computing $h(X)$ is NP-hard. Graphs with large $h(X)$ are useful in theoretical computer science (called expanders). However, there are eigenvalue bounds on $h(X)$.

Theorem. If $X$ is connected and $k$-regular, then $\frac{1}{2}(k-\theta_2) \leq h(X) \leq \sqrt{k^2-\theta_2^2}$.

For example, on $Q_d$, $k=d$ and $\theta_2 = d-2$ so $h(X) \geq 1$ and on $K_n$, $h(X) \geq \frac{n}{2}$.

Proof. Entire proof given in text. We prove only the lower bound. Consider the Laplacian $L$ of $X$. Recall that $z^TLz = z^T(\Delta - A)z = \sum_{i \sim j \in E(X)} (z_i - z_j)^2$. Assume the eigenvalues of $L$ are $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ So $\lambda_1 = 0$ and by connectivity $\lambda_2 > 0$. Notice that

$$\lambda_2 = \min_{\substack{z \neq 0 \\ 1^Tz = 0}} \frac{z^TLz}{z^Tz},$$

so we can obtain a bound for $\lambda_2$ by picking any $z$. Take $S \subseteq V(X)$ and take $z_u = n - |S|$ if $u \in S$ and $z_u = -|S|$ if $u \notin S$. Then

$$\lambda_2 \leq \frac{z^TLz}{z^Tz} = \frac{\sum_{i \sim j \in E(X)} (z_i - z_j)^2}{\sum_{i \in V(X)} z_i^2} = \frac{|\partial S|(n-s-(-s))^2}{s(n-s)^2 + (n-s)(-s)^2} = \frac{|\partial S| n^2}{s(n-s)n} = \frac{|\partial S|}{s(n-s)}$$

since the only terms contributing to the sum in the numerator are for those edges in $\partial S$.

If $|S| \leq \frac{n}{2}$ we get $\lambda_2 \leq \frac{2|\partial S|}{|S|}$, which implies $h(X) \geq \frac{1}{2}\lambda_2$. Then if $X$ is $k$-regular, $\lambda_2 = k - \theta_2$, so we get our bound. ▨

So to find graphs with large $h(X)$, we need graphs with large $k - \theta_2$.

Theorem (Alon-Boppana) If $\varepsilon > 0$ then there are only finitely many connected $k$-regular graphs $X$ such that $\theta_2 \leq \sqrt{2k-2} - \varepsilon$.

A graph is called Ramanujan if $|\theta_r| \leq \sqrt{2k-2}$ for all $r \geq 2$. Some examples are

Excluding the valency, every e/val $\in [-\sqrt{2k-2}, \sqrt{2k-2}]$. For bipartite, allow $\theta_{min}$ to be $-\theta_{max}$.

- → $K_n$ for $n \geq 3$,
- → Paley graphs, which are $(q, \frac{q-1}{2}; \frac{q-5}{4}, \frac{q-1}{4})$-strongly regular and so have eigenvalues $\frac{1}{2}(q-1)$ and $\frac{1}{2}(-1 \pm \sqrt{q})$
- → Latin square graphs, $(n^2, 3n-3; n, 6)$-SRGs
- → (bipartite) if $NN^T = nI - J$ then take $A = \begin{bmatrix} 0 & N \\ N^T & 0 \end{bmatrix}$, where $N$ is $(n^2+n+1) \times (n^2+n+1)$, $k = n+1$

e/vals of SRGs

if $X$ is $(n,k;a,c)$-SRG then e/vals are $k$ and roots of $t^2-(k-a)t-(k-c)$

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30 Mar 2015

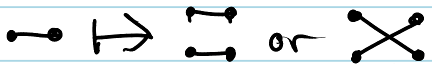
# Constructing Ramanujan Graphs

(a) Cayley graphs for $PSL(2,q)$

→ $GL(d,q)$ is the $d \times d$ matrices over $\mathbb{F}_q$
→ $SL(d,q) = \{x \in GL(d,q) \mid \det x = 1\}$
→ $PSL(d,q) = SL(d,q)/\{\text{scalar matrices}\}$

Finding the eigenvalues to confirm Ramanujan-ness is <u>hard</u>.

(b) bipartite Ramanujan graphs.

→ construct a two-fold cover / "2-lift" by taking  (for some unknown choice of parallel or crossing for each edge) of a smaller bipartite Ramanujan.

→ use the <u>matching polynomial</u>: a $k$-matching is a matching of size $k$ and say there are $p(X,k)$ $k$-matchings in $X$. The matching polynomial is

$$\mu(X,t) = \sum_k (-1)^k p(X,k) t^{n-2k}.$$

"tries hard to be the characteristic polynomial but fails in some significant ways"

We have some interesting recurrences. If $u \in V(X)$, $uv \in E(X)$, then

$$\mu(X,t) = \mu(X \setminus uv, t) - \mu(X \setminus \{u,v\}, t), \text{ and}$$

$$\mu(X,t) = t\mu(X \setminus u, t) - \sum_{v \sim u} \mu(X \setminus \{u,v\}, t)$$

From the latter we immediately see $\mu(K_{n+1},t) = t\mu(K_n,t) - n\mu(K_{n-1},t)$ and $\mu(P_{n+1},t) = t\mu(P_n,t) - \mu(P_{n-1},t)$. We also have that

$$\mu(X \sqcup Y, t) = \mu(X,t)\mu(Y,t) \quad \text{and} \quad \frac{d}{dt}\mu(X,t) = \sum_{u \in X} \mu(X \setminus u, t).$$

We introduce <u>signed graphs</u>. Given a map $\sigma : E(X) \to \{\pm 1\}$ we get a signed graph $X^\sigma$ (not to be confused with oriented graph) with adjacency matrix $A^\sigma$ s.t. $A^\sigma_{uv} = \sigma(uv)$ if $u \sim v$, else $0$. ($A^\sigma$ is symmetric)

<u>Theorem (Gutman)</u> $\mu(X,t) = 2^{-|E(X)|} \sum_\sigma \phi(X^\sigma, t)$.

<u>Claim.</u> All signings of a tree have the same characteristic polynomial.

<u>Corollary.</u> If $X$ is a forest, $\phi(X,t) = \mu(X,t)$. <u>Thm?</u> If and only if.

<u>Theorem</u> ?

perfect matchings on $X$?

<u>Proof.</u> Sketched. It is enough to show that $pm(X)$ is the average of $\det(A^\sigma)$ over all signings $\sigma$. Recalling that $\det(A) = \sum_{P \in Sym(n)} \det(A \circ P)$, we note that if $P$ has a fixed point then $\det(A \circ P) = 0$. Then

Recall the Schur product $M \circ N$, defined by point-wise multiplication

$$\sum_\sigma \det(A^\sigma) = \sum_\sigma \sum_P \det(A^\sigma \circ P) = \sum_P \sum_\sigma \det(A^\sigma \circ P) = \sum_P \sum_\sigma \det((A \circ P)^\sigma),$$

and claim the summands corresponding to $P$ not involutions vanish.

CO 444

1 April 2015

# Continued From Yesterday

Let $E=|E(X)|$ and recall $pm(X)=2^{-E}\sum_{\sigma, P} \det(A^{\sigma} \circ P)$ and $\mu(X,t)=2^{-E}\sum_{\sigma} \phi(X,t)$. We give a second proof, using the edge-deletion recurrence. Divide up $Sym(V(X))$ into

(a) permutations not "using" (12) or (21) $\to$ $pm(X \setminus 1\sim 2)$
(b) permutations using (12) as a cycle
(c) permutations using (12) but not as a cycle $\to$ contributes zero?

Something something idk. ┐(°_o)┌

## Double Covers (2-lifts)

Fig. 1

We have a bijection between signings of X and double-covers: send (+) to parallel edges and (−) to crossing edges. In terms of adjacency matrices, $A(\hat{X})$ is found by taking 0 to $\begin{bmatrix}0&0\\0&0\end{bmatrix}$, 1 to $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and −1 to $\begin{bmatrix}0&1\\1&0\end{bmatrix}$. These matrices are simultaneously diagonalizable by $H=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$, and we see

$$H\begin{bmatrix}0&0\\0&0\end{bmatrix}H=\begin{bmatrix}0&0\\0&0\end{bmatrix},\quad H\begin{bmatrix}1&0\\0&1\end{bmatrix}H=\begin{bmatrix}1&0\\0&1\end{bmatrix},\text{ and }\quad H\begin{bmatrix}0&1\\1&0\end{bmatrix}H=\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

So then $\begin{bmatrix}H&&\\&\ddots&\\&&H\end{bmatrix}A(\hat{X})\begin{bmatrix}H&&\\&\ddots&\\&&H\end{bmatrix}$ gets you something like $A(X^{\sigma})$ but with $0\mapsto\begin{bmatrix}0&0\\0&0\end{bmatrix}$, $1\mapsto\begin{bmatrix}1&0\\0&1\end{bmatrix}$, and $-1\mapsto\begin{bmatrix}1&0\\0&-1\end{bmatrix}$. This is permutationally equivalent to $\begin{bmatrix}A(X)&0\\0&A(X^{\sigma})\end{bmatrix}$ so we have that $\phi(\hat{X},t)=\phi(X,t)\phi(X^{\sigma},t)$. This is how you show that there exists a signing giving rise to a Ramanujan double-cover of a Ramanujan.

## Path-Trees

The vertices of a path-tree of X are the paths of X starting at some distinguished vertex. Two paths are adjacent if one is a maximal subpath of the other.

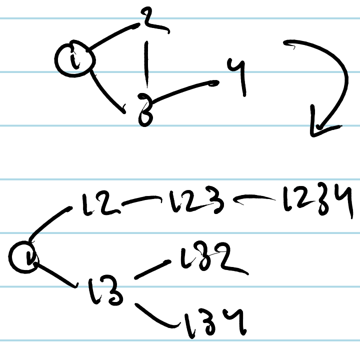


Fig. 2

**Theorem.** If $a \in V(X)$ and T is the path-tree of a, (?)

$$\frac{\mu(X\setminus a,t)}{\mu(X,t)}=\frac{\mu(T\setminus a,t)}{\mu(T,t)}=\frac{\phi(T\setminus a,t)}{\phi(T,t)}$$

and $\mu(X,t) \mid \mu(T,t)$.

**Proof.**

The claim is that the two graphs at right have the same matching polynomial, so then the ratios with those deleting the vertex 1 is also equal. Notice in the second, 1 is a cut vertex.

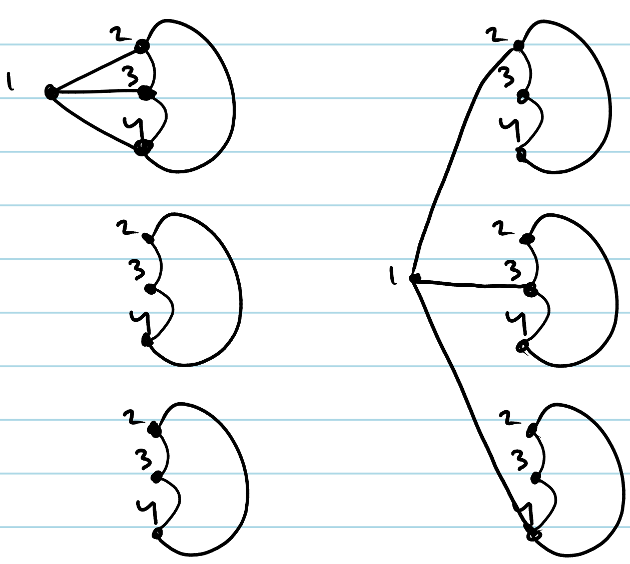


Fig. 3

$$\mu(X)=t\mu(X\setminus 1)-\mu(X\setminus 12)-\mu(X\setminus 13)-\mu(X\setminus 14)$$

$$\mu(\text{left})=\mu(X)\,\mu(X\setminus 1)$$

$$\mu(\text{right})=t\mu(X\setminus 1)^3-\mu(X\setminus 12)\mu(X\setminus 1)^2-\mu(X\setminus 13)\mu(X\setminus 1)^2-\mu(X\setminus 14)\mu(X\setminus 1)^2=\mu(\text{left}).$$

Also $\frac{\mu(X\setminus 1)}{\mu(X)}=\frac{\mu(\text{left}\setminus 1)}{\mu(\text{left})}=\frac{\mu(\text{right}\setminus 1)}{\mu(\text{right})}$ and then inductively wave hands. ▨

CO 444

2 Apr 2015

# Eigenstuff Again Maybe

If $Az=\lambda z$ then $|\lambda| \le \max_i \sum_j |A_{ij}|$ which is the maximum valency if A is an adjacency matrix. To see this $\lambda z_u = \sum z_v$ and then $|\lambda||z_u| \le \sum_{v\sim u} |z_v|$. Let $|z|$ be the vector such that $|z|_i = |z_i|$. Then

$$|\lambda||z| \le A|z| \Rightarrow |\lambda||z_{\max}| \le \max_i \Big\{\sum_j |A_{ij}|\Big\} |z_{\max}| \Rightarrow |\lambda| \le \max_i \sum_j |A_{ij}|.$$

Consider again the k-regular tree (Fig. 1). Taking the distance partition we find it is equitable and the quotient graph is fairly small and simple. The eigenvalues are still real on the quotient, and the corresponding eigenvectors are constant on the cells of the partition. The eigenvectors we lose are orthogonal to those, so they sum to zero on each cell of the partition.

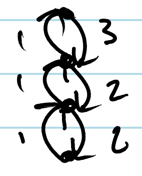


Fig. 2

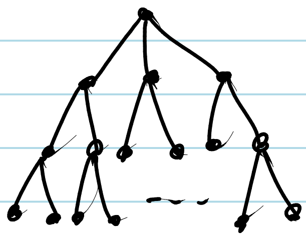


Fig. 1 $k=3$

So what are the eigenvalues of the quotient. In this example, the adjacency matrix is $\begin{bmatrix} 0&3&0&0\\ 1&0&2&0\\ 0&1&0&2\\ 0&0&1&0 \end{bmatrix}$, which is big so let's go down one level: $\begin{bmatrix} 0&3&0\\ 1&0&2\\ 0&1&0 \end{bmatrix}$. Multiplying on the left of this matrix by D diagonal and on the right by $D^{-1}$ preserves eigenvalues, so

$$\begin{bmatrix} a&&\\ &b&\\ &&c \end{bmatrix}\begin{bmatrix} 0&3&0\\ 1&0&2\\ 0&1&0 \end{bmatrix}\begin{bmatrix} a^{-1}&&\\ &b^{-1}&\\ &&c^{-1} \end{bmatrix} = \begin{bmatrix} 0 & 3\frac{a}{b} & 0\\ \frac{b}{a} & 0 & 2\frac{b}{c}\\ 0 & \frac{c}{b} & 0 \end{bmatrix}$$

and this extends the obvious way to larger such quotients. With some "fiddling", we find a bound of $2\sqrt{k-1}$ on the spectral radius, meaning the spectrum lands in $(-2\sqrt{k-1}, 2\sqrt{k-1})$ and we have fun expander times all around.

This sort of matrix is called <u>tridiagonal</u> — nonzero only on the main diagonal, the first superdiagonal, and the first subdiagonal, and also has the products of corresponding off-diagonal entries nonnegative. We look at the characteristic polynomial.

$$\det\begin{bmatrix} t&-3&0&0\\ -1&t&-2&0\\ 0&-1&t&-2\\ 0&0&-1&t \end{bmatrix} = t\det\begin{bmatrix} t&-2&0\\ -1&t&-2\\ 0&-1&t \end{bmatrix} + 3\det\begin{bmatrix} -1&-2&0\\ 0&t&-2\\ 0&-1&t \end{bmatrix} = t\det\begin{bmatrix} t&-2&0\\ -1&t&-2\\ 0&-1&t \end{bmatrix} - 3\det\begin{bmatrix} t&-2\\ -1&t \end{bmatrix}.$$

<u>Claim.</u> The determinants of these tridiagonal matrices form a <u>3-term recurrence</u> — in this case, $f_0=1$, $f_1=t$, $f_2=t^3-2$, and

$$f_{m+1} = tf_m - (k-1)f_{m-1} \quad \text{for } m \ge 2 \text{ where } k=3.$$

The interesting thing is that tridiagonals are diagonally similar to symmetric matrices, so they have real roots.

We also have that these polynomials are orthogonal with respect to an inner product. "E.g."

$$\langle p,q\rangle = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} p(t)q(t)e^{-t^2/2}\,dt$$

Suppose that $f_0, f_1, f_2, \ldots$ are the polynomials obtained by performing Gram-Schmidt on $1, t, t^2, \ldots$ They are orthogonal and we scale them to be monic. Multiplication by $t$ is self-adjoint with this inner product so $\langle tf_m, f_n\rangle = \langle f_m, tf_n\rangle$, and this is zero if $m \ge n+2$ or $m \le n-2$. So $tf_n = b_n f_{n-1} + a_n f_n + c_n f_{n+1}$ or something, modulo off-by-one errors. Also

$$\begin{bmatrix} 1 & t & t^2-3 & t^3-5t & \cdots \end{bmatrix}\begin{bmatrix} 0&3&&\\ 1&0&2&\\ &1&0&2\\ &&1&\ddots \end{bmatrix} = t\begin{bmatrix} 1 & t & t^2-3 & t^3-5t & \cdots \end{bmatrix}$$

provided $t$ is an e'val of the tree (so the proper thing zeroes in the end).

— — * — —

And that's AGT. I can be reached by email at ischtche@uwaterloo.ca. Keep an eye out for the TeXed notes at <u>this URL</u> in the (near?) future.

http://csclub.uwaterloo.ca/~ischtche/co444.pdf