PMC SASMS: Nondistributive lattices

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At the behest of Sean Harrap, I am going to drown you all in abstract algebra. This is my revenge against Prof. David DeVidi and PHIL 371 *Intro to substructural logics*.

Let $(L, \le, \wedge, \vee)$ be a lattice. This means that $(L, \le)$ is a poset, and also ($x \le a \wedge b$ iff $x \le a$ and $x \le b$) and ($x \ge a \vee b$ iff $x \ge a$ and $x \ge b$). Lattices are pretty cool and show up in a lot of theory. Here are some examples.

— (powerset, $\subseteq, \cap, \cup$). — $(\mathbb{N}, \text{divides}, \gcd, \text{lcm})$.

— (totally ordered set, $\le, \min, \max$).

— (subgroups of some group $G, \subseteq, \cap, \langle\cup\rangle$).

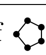
Now let's get used to the arithmetic of lattices. First off, if $x \le y$, then $x \wedge y = x$ and $x \vee y = y$. Consequently, since $a \le a \vee b$, we have $a \wedge (a \vee b) = a$.

Some more. We have that $a \wedge b \le a$ and $a \wedge b \le b \le b \vee c$, so we derive $a \wedge b \le a \wedge (b \vee c)$. By symmetry in b and c,

$$(a \wedge b) \vee (a \wedge c) \le a \wedge (b \vee c).$$

This equation is called **weak distributivity**—$\wedge$ is almost distributing over $\vee$. If $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$ always, L is called **distributive**. My first three examples are distributive, but the last is not in general. E.g. $G = Q_8$.

What I will talk about today is a Kuratowski–like *excluded subobject* theorem about distributivity. Specifically,

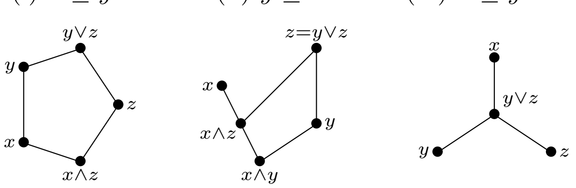
Thm. Every nondistributive lattice contains one of  or  as a sublattice.

To do that, we will examine 'tuples' $(x; y, z)$ that *violate distributivity*, i.e. $x \wedge (y \vee z) > (x \wedge y) \vee (x \wedge z)$. First off, let's learn about how x and y and z are positioned in $(L, \le)$.

Prop. If $x \wedge (y \vee z) > (x \wedge y) \vee (x \wedge z)$, then x, y, z are mutually incomparable, except possibly WLOG $x > z$.

Pf. Suppose for a contradiction, in three cases:

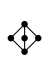
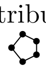
(i) $x \le y$ (ii) $y \le z$ (iii) $x \ge y \vee z$

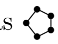


If only one of $x \ge y$ or $x \ge z$ holds, that's okay. □

As it turns out, a violation of distributivity with $x \ge z$ is especially bad. Such a triple violates even the *modular law*:

$$\text{if } x \ge z \text{ then } x \wedge (y \vee z) = (x \wedge y) \vee z.$$

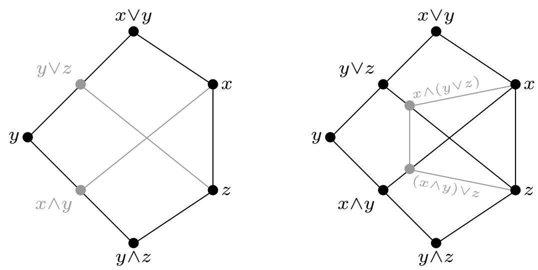
Distributivity implies modularity, and that  is modular but  is not. This suggests we split our theorem in two:

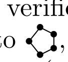
$\frac{1}{2}$**Thm.** Every nonmodular lattice has  as a sublattice.

$\frac{1}{2}$**Thm.** Every nondistributive modular lattice contains a copy of  as a sublattice.

The first half is easier than the second, so let's do that.

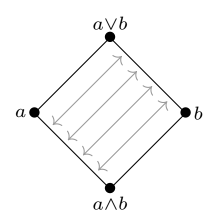
Pf. $(\frac{1}{2})$ Suppose there exist $x, y, z \in L$ such that $x > z$ and $x \wedge (y \vee z) > (x \wedge y) \vee z$. Then consider this sublattice:



It is readily verified that the inner pentagon is a lattice isomorphic to , using the Proposition from earlier. For example, if $x \wedge (y \vee z) < y \vee z$, or else $x \ge y \vee z \ge y$. □

From now on, we will assume L is modular. To prove the second $\frac{1}{2}$-theorem, we're going to need some lemmas. This is where things will get computational: the faint of heart may wish to leave the room.

Diamond Isomorphism Principle. The function $x \mapsto x \vee b$ is an order-preserving bijection between the intervals $\frac{a}{a \wedge b}$ and $\frac{a \vee b}{b}$, with inverse $y \mapsto y \wedge a$.



Pf. If $a \wedge b \le x \le a$, then $b = b \vee (a \wedge b) \le b \vee x \le a \vee b$. Then we find that $a \wedge b \le a \wedge (b \vee x) \le a \wedge (a \vee b) = a$. But since $a \ge x$, we have $a \wedge (b \vee x) = (a \wedge b) \vee x = x$ by modularity. So the two maps are mutually inverse order-preserving maps. □

This lemma tells us that, if we have some inequality, then we can hit both sides with sensible meets and joins and still have an inequality on our hands.

So now define operations

$$\begin{aligned}
\lfloor a, b, c \rfloor &= (a \wedge b) \vee (b \wedge c) \vee (c \wedge a), \\
\lceil a, b, c \rceil &= (a \vee b) \wedge (b \vee c) \wedge (c \vee a), \\
m(a, b; c) &= (a \vee b) \wedge \big(c \vee (a \wedge b)\big) \\
&= \big((a \vee b) \wedge c\big) \vee (a \wedge b).
\end{aligned}$$

The function m is unambiguous, by modularity. Observe that $\lfloor a,b,c\rfloor \le m(a,b;c) \le \lceil a,b,c\rceil$ by weak distributivity and weak dual distributivity:

$$\begin{aligned}\lfloor a,b,c\rfloor &= ((a\wedge c)\vee(b\wedge c))\vee(a\wedge b)\\ &\le ((a\vee b)\wedge c)\vee(a\wedge b) = m(a,b;c)\\ &= (a\vee b)\wedge(c\vee(a\wedge b))\\ &\le (a\vee b)\wedge((c\vee a)\wedge(c\vee b)) = \lceil a,b,c\rceil.\end{aligned}$$

These will be our candidates for the copy of ◈. So for that purpose, we need to know that none of them are equal to each other.

Lemma. If $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$, then $\wedge$ and $\vee$ distribute over each other in $\{a,b,c\}$.

Pf. (Sketch) Apply modularity to *weak dual distributivity*, which is $(a\wedge b)\vee c \le (a\vee c)\wedge(b\vee c)$, around a.

$$\begin{aligned}(a\wedge b)\vee c &= (a\wedge b)\vee(a\wedge c)\vee(a\wedge b)\vee c\\ &= \big(a\wedge(b\vee c)\big)\vee(a\wedge b)\vee c\\ &= \big((a\vee c)\wedge(b\vee c)\wedge a\big)\vee(a\wedge b)\vee c\\ &= (a\vee c)\wedge(b\vee c)\wedge\big(a\vee(a\wedge b)\vee c\big)\\ &= (a\vee c)\wedge(b\vee c)\wedge(a\vee c)\\ &= (a\vee c)\wedge(b\vee c).\end{aligned}$$

From $D(a;b,c)$ we deduce $D^*(c;a,b)$. Apply this argument repeatedly to get $D(b;c,a)$, etc. □

This lemma tells us that if we violate distributivity in $\{a,b,c\}$ even once, then we violate it a lot. It follows that our five elements are distinct.

Now we can go about proving the theorem.

Pf. $(\frac{2}{2})$ Suppose there exist $x,y,z\in L$ violating the distributive law. Then $\lfloor x,y,z\rfloor < m(x,y;z) < \lceil x,y,z\rceil$ by the Lemma and the Diamond Isomorphism Principle. Let

$$X = m(y,z;x),\quad Y = m(z,x;y),\quad Z = m(x,y;z).$$

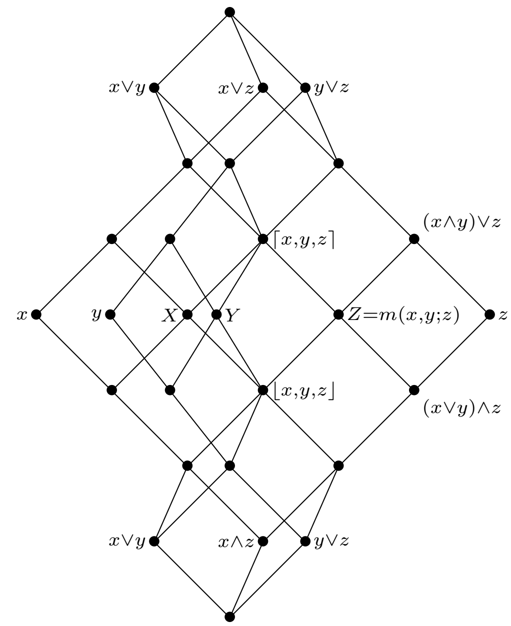
I claim that $\{X,Y,Z,\lfloor x,y,z\rfloor,\lceil x,y,z\rceil\}$ is a sublattice isomorphic to ◈. Well, we can compute that

$$\begin{aligned}Z\vee X &= m(x,y;z)\vee m(y,z;x)\\ &= ((x\vee y)\wedge z)\vee(x\wedge y)\vee((y\vee z)\wedge x)\vee(y\wedge z)\\ &= ((x\vee y)\wedge z)\vee(x\wedge(y\vee z))\vee(x\wedge y)\vee(y\wedge z)\\ &= \big(((x\vee y)\wedge z)\vee x)\wedge(y\vee z)\big)\vee(x\wedge y)\vee(y\wedge z)\\ &= \big((x\vee y)\wedge(z\vee x)\wedge(y\vee z)\big)\vee(x\wedge y)\vee(y\wedge z)\\ &= \lceil x,y,z\rceil\vee(x\wedge y)\vee(y\wedge z) = \lceil x,y,z\rceil,\end{aligned}$$

so by symmetry and duality, we're done. □

Eugh. That was disgusting. Can I get a picture?

Well, let me try to draw you a picture. In the year 1900, Richard Dedekind published a paper introducing the modular law for lattices and computing the *free modular lattice on three generators*, FML_3.



This is a very spooky object and it has many spicy properties but what is important for us is that because it is free, proving that FML_3 has a copy of ◈ in it is *almost* as good as proving that second $\frac{1}{2}$-theorem.

The only thing that remains to be seen is that nothing collapses when you map x,y,z to a triple a,b,c violating distributivity. This can be accomplished with some meditation upon the Diamond Isomorphism Principle, but is essentially equivalent in difficulty to the proof I have detailed. You are merely offloading some of the computation to Dedekind.

References

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— Birkhoff. *Lattice Theory.* Providence: AMS, 1973. ISBN 978-0821810255.

— Dedekind. *Ueber die von drei Moduln erzeugte Dualgruppe.* Mathematische Annalen Bd. 53 (1900) pp. 371–403. `http://www.digizeitschriften.de/dms/resolveppn/?PID=GDZPPN002257947`